Mathematical Models of Target Coverage and Missile Allocation

A. ROSS ECKLER

STEFAN A. BURR

PUBLISHED BY

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HISTORICAL NOTE

The art of projecting missiles is very old, dating back at least to the Roman ballista, but it was placed on a scientific footing until the sixteenth century, when the Italian mathematician Niccolo Fontana Tartaglia studied the trajectories of missiles fired from weapons ranging from pistols to cannon. As the first mathematician to optimize the aim of a weapon one might call him the prototype of the modern missile analyst. Yet his knowledge was purchased at a price, as revealed by the following passage toward the end of the dedication in his Nova Scientia Inventa (1537):

> "But then in reflecting one day it struck me as blameworthy, infamous, and cruel, and meriting no small punishment before God, to wish to refine an art so injurious to one's fellow men — a vile destroyer of the human race, and especially of Christians in their incessant warfare."

Similar misgivings about the social consequences of scientific work devoted to war have been expressed ever since, culminating in the angst of the atomic scientists after World War II. The authors of this monograph are not immune; but our concern has been tempered by the hope that a quantitative understanding of missile defense strategies may actually reduce the probability of international conflict. At the very least, this monograph should discourage any naive belief that a perfect defense is possible.

FOREWORD

The publication of this monograph represents a new venture in our continuing effort to broaden the services that the Military Operations Research Society (MORS) offers to the professional military operations research analyst. It is our hope that the response generated by this publication will encourage a continuing series of monographs of special interest to our society. In particular, we place high value on the encouragement to authors that such a series might offer and the consequent enlargement of the literature of military operations research.

The MORS is extremely fortunate and proud to have Drs. Eckler and Burr's monograph as our first publication. A first publication always sets a standard for others to follow. As such, this monograph represents the highest standards of both technical excellence and relevance to military operations research.

I also want to recognize the MORS committee on publication and its Chairman, Mr. Sid Moglewer, who conceived of this project and carried it through to the very successful conclusion. Particular recognition should go to Dr. Walter Deemer, who as a member of that Committee identified the original manuscript and gave generously of his time to its publication.

ROBERT H. STEVENS
President 1971-72

PREFACE

It is commonplace for the authors of a survey monograph to invite readers to submit additions or corrections for a possible later edition. We are keenly aware that we are guilty of sins of omission, for the literature on the target coverage and missile allocation problem is widely dispersed, and much of it is virtually inaccessible to the layman. We are interested in giving credit for priority associated with each methodology; however, our major interest is not in tracing the historical thread of a development but in making sure that the most important ideas have been brought together and systematically compared.

In a book with two authors, questions inevitably arise concerning the nature of each one's contributions. The senior author (A. R. Eckler) has been responsible for searching the literature for relevant material, for deciding upon the basic structure of the book, and (for the most part) for writing up results in a form intelligible to the non-mathematical reader. The junior author (S. A. Burr) has been responsible for correcting, clarifying and occasionally developing in detail the mathematics, as well as improving the organization and exposition of the monograph in many sections. This division of responsibility may help the reader decide to whom any criticisms, additions or inquiries should be addressed.

Many sections of this monograph were originally developed by Bell Telephone Laboratories colleagues of the authors during the years 1965 through 1970; their work has materially enhanced the scope of this book One of the motivations for writing this book was to bring their excellent work to the attention of a wider audience. These contributors were:

D. J. Brown*	J. A. Hooke	M. J. Spahn*
J. Eilbott	S. Horing	C. W. Spofford
M. L. Eubanks*	D. Jagerman	R. E. Thomas
D. Guthrie*	H. Polowy*	M. S. Waterman*
H. Heffes	W. L. Roach	F. M. Worthington*
	S. A. Smith	

An asterisk after the name indicates that the author is no longer associated with Bell Telephone Laboratories. It is hoped that sufficient details of their work have been given to satisfy the needs of most readers interested in missile allocation strategies. However, the occasional reader who requires more detailed information about these models may telephone the senior author at Bell Telephone Laboratories, Holmdel, New Jersey.

May 7, 1972

A. Ross Eckler
Stefan A. Burr

TABLE OF CONTENTS

CHAPTER ONE

AN OUTLINE OF OBJECTIVES AND SOME UNDERLYING ASSUMPTIONS

During the past twenty years a large number of mathematical investigations have been carried out in two broad areas: (1) properties of attacks on point and area targets by weapons having various aiming-errors and destruction capabilities; (2) allocation of missiles for the defense of a group of point targets, and allocations of weapons for attacks against a group of point targets. Unfortunately, these investigations are so varied and so widely scattered that it is difficult to obtain a unified view of the subject. Some are found in the classified literature; others are reported on in government and company documents; many have been published in the Annals of Mathematical Statistics, Journal of the American Statistical Association, Technometrics, Biometrika, Journal of the Operations Research Society of America, SIAM Review, American Mathematical Monthly, and similar journals. The object of this monograph is to bring together these investigations and present them in a more or less logical pattern.

None of the references actually cited in this monograph is classified. Perhaps surprisingly, it has not been found essential to cite any. Mathematical models, by their nature, do not normally require classification, and it appears that the authors of classified studies have usually tried to publish the mathematical aspects of such studies in the unclassified literature. It is hoped that this monograph may serve to encourage this practice.

Although all references are unclassified, many (particularly government and company documents alluded to previously) are not easily obtainable. Many documents are available through the Defense Documentation Center (DDC), Cameron Station, Alexandria, Va. The general difficulty of obtaining such information was one of the motivations for writing this book.

This monograph is directed toward two different classes of readers. The more important reader is the engineer who is charged with the responsibility of designing a rational missile defense of a group of targets threatened by an offensive weapon force (such as intercontinental ballistic missiles, or shorter-range submarine-launched missiles). Generally he has neither the time nor the inclination to wade through detailed mathematical proofs in order to extract the essential ideas in a potentially relevant article. Accordingly, this monograph omits most mathematical proofs except those of a trivial nature; the reader interested in proofs must generally examine the original references. The few exceptions to this policy generally represent cases where no adequate proof exists in the literature, and even then proofs are often only sketched. Instead, great emphasis has been placed on displaying the underlying assumptions of each model, comparing or contrasting it with other models.

This monograph is also directed toward mathematical analysts, particularly game theorists and probability theorists. It is clear that there are many challenging unsolved problems in missile offense and defense. In various places in the text, conjectures are stated; sometimes, these are supported by numerical examples or by Monte Carlo simulation. In other places, the omission of certain topics suggests areas of further research. For example, damage assessment strategies are discussed when the targets being defended have the same value, but no mention of this topic is made when considering targets of unequal value.

This monograph may interest those concerned with operations research in general. Even though the models in this monograph are idealized, for the most part they have arisen from attempts to apply mathematics to real problems. There is something to be learned from examining the compromises between realism and mathematical tractability that have been made.

Finally, some of the models considered may be applicable to topics other than missile allocation. In fact, a number of the references in this monograph use either non-military examples, or are formulated abstractly. For the sake of exposition, such references are often presented here in a manner that may suggest the author had missile allocation in mind when in fact such was not the case. Perhaps the most striking example of this is Kabak's paper at the end of Section 3.6; he applied the mathematical model to the scheduling of the delivery of babies. Further, the reader is forewarned that minor changes and corrections have been made (usually without comment) in the formulations and solutions of the models of the references. However, these changes do not affect the basic results of the original work.

In many missile offense and defense studies (particularly those which appear in the classified literature), a large number of detailed assumptions are made in order to mirror reality as closely as possible. The resultant mathematical model is usually so complicated that analytic solutions are impossible. Typically, it then becomes necessary to resort to Monte Carlo simulation on a digital computer in order to evaluate the results of strategies arrived at externally to the model. Simulation can rarely generate enough different results to find an approximate optimum with high confidence. This monograph, in contrast, attempts to dissect these complex models into their component parts, examining the effects of assumptions taken a few at a time. By doing this, an analytic formulation is often possible, and one can better understand the effects of each assumption upon the final answer. In short, it is unlikely that the models presented in this monograph can be used directly to solve realistic missile offense and defense problems, but it is hoped that they may provide insight into such problems and may be used as part of a practical solution. At the very least, the missile analyst should gain some idea of the likelihood that an analytic approach to this problem would be fruitful.

1.1 THE CHOICE OF A CRITERION OF EFFECTIVENESS

If the defense knows the size of the offensive stockpile to be used against a set of targets, and the offense knows the size of the defensive stockpile, the criterion of effectiveness generally used is that of expected target value destroyed (or saved). The defense wishes to select strategies which minimize this quantity, whereas the offense wishes to select strategies which maximize this quantity. One speaks of expected target value destroyed, rather than actual target value destroyed, because of the typical uncertainties of the outcome of the engagement; for example, it is usually assumed that an unintercepted offensive weapon has a probability p of destroying the target against which it is directed, and a defensive missile has a probability ρ of destroying the offensive weapon it is directed at. Furthermore, in multiple target situations, the actual number of defensive missiles and offensive weapons associated with each target is likely to be unknown to the defense.

Although the criterion of expected target value destroyed has the great virtue of mathematical tractability, it may not be the most suitable criterion in a specific missile offense or defense problem. For example, the defense may be much more anxious to know how much target value will survive (say) 90 per cent of all attacks of a given size. Ordinarily, criteria such as these are much more difficult to deal with analytically; however, they can easily be used in Monte Carlo simulations. Instead of determining the probability density function of surviving target value, one might perhaps seek to calculate the mean and variance of this density.

Instead of minimizing the expected target value destroyed, the defense can elect to maximize the probability that no target value is destroyed. This alternative criterion is particularly appealing when the number of defensive missiles available at a target (or group of targets) equals or exceeds the number of offensive weapons directed at that target (or group of targets). This might be the situation in an accidental attack; however, it seems plausible that a determined and rational offense will not normally attack a target unless he can count on fairly heavy damage. Analogously, the offense can elect to maximize the probability that all target value (or a predetermined function of target value) is destroyed, if surviving targets are intolerable from his standpoint.

It is likely that the choice of a criterion will depend more strongly upon the nature of the target than on the relative numbers of defensive missiles and offensive weapons. It seems reasonable to use the expected target value destroyed criterion for targets such as buried silos containing ICBMs; the usefulness of undamaged targets does not depend upon the fraction of targets destroyed. However, if one considers targets such as cities, it may happen that even a relatively small amount of damage will have almost as catastrophic effects as a large amount of damage. (Penetrators after the first contribute very little more to the general catastrophe.) If this is so, a defense strategy which maximizes the probability of no damage (even though the probability of no damage is rather low) may be the rational one to choose. These are difficult and controversial choices, and the reader will not find the answers to them in this monograph.

A third criterion of effectiveness is defined if one minimizes the expected number of weapons not intercepted by the defense. This asymptotically approaches the first criterion when the probability of target destruction (or the expected fraction of the target destroyed) by an unintercepted weapon approaches zero. There exist at least three significant cases in which the second and third criteria lead to identical defense strategies, and many in which they differ. It would be of considerable interest to specify the precise range of conditions under which different criteria lead to the same strategy. Three cases where they do lead to the same strategy are:

1. A single target is attacked by A weapons and defended by D missiles. Each missile destroys the weapon at which it is aimed with probability ρ; missile engagement outcomes are independent of each other. Target damage is proportional to the number of weapons which are not destroyed (in other words, the expected target value destroyed is proportional to the expected number of weapons which are not destroyed).

2. A set of T equal-valued targets is attacked by one offensive weapon apiece, and defended by D missiles. Each missile destroys the weapon at which it is aimed with probability ρ; missile engagement outcomes are independent of each other. The probability that the ith target is destroyed if the weapon directed at it is not intercepted is permitted to depend on i.

3. A single target is attacked by A weapons and defended by D missiles. Each missile destroys the weapon at which it is aimed with probability ρ; missile engagement outcomes are independent of each other. The defense uses a two-stage shoot-look-shoot strategy; that is, it allocates D-m missiles to the attackers, observes which attackers survive this defense, and then allocates m missiles to the survivors. Target damage is proportional to the number of weapons which are not destroyed at either stage (in other words, the expected target value destroyed is proportional to the expected number of weapons which are not destroyed).

However, the two criteria lead to different defensive allocations in the following single-target situation. Assume that the defense has D missiles and the offense has A weapons, but that the actual attack size is given by the probability density function Pr (i attackers) = p_i, $i = 1, \ldots, A$, $\Sigma p_i = 1$. The probabilities p_i are assumed to be known by the defense. A simple numerical example will illustrate that different defense strategies are required. Assume that A = D = 3, and that $p_1 = 0.3$, $p_2 = 0.1$ and $p_3 = 0.6$. Assume also that the defensive missile reliability ρ is equal to 0.5. The defensive strategy minimizing the expected number of penetrators consists of allocating one defensive missile to each offensive weapon that arrives; the expected number of penetrators is equal to 1.15, and the probability of no penetrators is 0.250. The defensive strategy maximizing the probability of no penetrators consists of allocating two missiles to the first arriving weapon and one missile to the second arriving weapon; the expected number of penetrators is 1.30 and the probability of no penetrators is 0.265. As in the above example, however, the two criteria lead to fairly similar results.

If the defense has no information about the size of the offensive stockpile, it is not possible to design a strategy which minimizes the expected fraction of targets destroyed (or maximizes the probability that no targets are destroyed). However, other criteria can be formulated. If the offense fires weapons one at a time against a single target, one natural objective for the defense is to maximize the expected number of weapons until the first penetrator (that weapon which destroys the target).

Alternatively, the defense can design its strategy so that the expected payoff per offensive weapon is as small as possible. To do this, the defense selects a strategy so that the expected fraction of targets destroyed (or, for a single target, the probability of target destruction) is essentially proportional to the attack size. Obviously, it is not possible to maintain proportionality beyond that attack size which exhausts the defensive stockpile.

Let A denote the attack size which exhausts the defense, and let F denote the expected fraction of targets destroyed at that time. For attacks of size $a \geq A$, and under appropriate assumptions, the expected fraction of targets destroyed will follow an exponential law:

$$E = F + (1 - F)\left(1 - \exp\left(-\alpha\,(a-A)\right)\right),$$

where α is a measure of target hardness and offensive weapon yield and accuracy. The object of the defense is to select the pair (F,A) so that

$$F/A = \max_{a \geq A} E/a\ .$$

It is not hard to show that if the pair (F,A) is selected so that $F \geq \alpha A/(1 + \alpha A)$, then for all $a > A$, $dE/da < F/A$, and the defense objective is assured.

Other criteria of optimality can be proposed. What seems to be needed is an underlying logical framework within which these and other criteria might be placed and compared. What is a natural criterion to choose in a given defense situation? If (as appears to be the situation) the criterion is related to the degree of knowledge each side has about the other, such a choice may have to depend heavily on intuition.

As a final comment, it should be emphasized that the choice of a criterion is usually somewhat arbitrary and its definition is usually based on imprecise data. For this reason, it may often be appropriate to choose a criterion for its mathematical tractability, rather than for its closeness to some possibly arbitrary objective.

1.2 SOME COMMENTS ON THE SCOPE OF THE MONOGRAPH

No attempt has been made to formulate and compare mathematical models for all aspects of missile offense and defense; this section briefly describes the scope and limitations of this work.

To begin with, questions involving costs or economic use of limited resources have been ignored (with the sole exception of Sections 6.3 - 6.3.5). The mathematical models in Chapters 4 and 5 usually begin with assumptions about offensive weapon and defensive missile stockpile sizes, and the probabilities that a target survives an attack by either an intercepted or unintercepted weapon. To calculate such probabilities, it is necessary to know such parameters as the reliability of the defensive missile, the yield of the offensive weapon, the aiming-accuracy of the offensive weapon, and the hardness of the target. In Chapter 2 the aiming-accuracy is included in the mathematical model, but the yield of the weapon and hardness of the target are lumped together in a single quantity (the radius of effectiveness of the weapon). However, parameters such as these are not fixed quantities given to the missile defense engineer; many different possibilities can be considered. For example, a budget-constrained offense can manipulate weapon yield, stockpile size and aiming-accuracy; similarly, a budget-constrained defense can spend money either on hardening targets or procuring more (or more reliable) defensive missiles. Economic choices are usually difficult to formulate in realistic mathematical models; often, the missile defense engineer is reduced to proposing various equal-cost systems and evaluating the performance of each one. The missile allocation models of this monograph may be helpful in designing a system, but they can do only part of the job.

In Sections 6.1.1 - 6.1.6 models are considered in which it is assumed that the offense overwhelms the defense by means of radar destruction, defensive stockpile exhaustion, or offensive leakage (offensive weapons penetrating when defensive missiles assigned to them fail to destroy them). However, the offense has other options, such as the use of concealment (blackout of radars by means of chaff or atmospheric ionization) or active jammers. The mathematical models in this monograph do not take tactics such as this into account; in effect, perfect radars and adequate data-processing facilities are assumed. (The only exception is in Section 3.5.2, where it is assumed that the radar requires T seconds to process each intercept.)

The only decoy model discussed in the monograph can be found in Section 3.4.0. It is quite difficult to specify in a mathematical model the changing visibility of a decoy as it approaches the target. Also, it is not certain that the offense will want to use decoys at all; he may prefer to use many weapons of relatively low yield, counting on leakage or defensive stockpile exhaustion to destroy the target.

With so many topics excluded, one might plausibly conclude that there remains little for this monograph to discuss. Yet this is far from the case; coverage and allocation problems reveal an unsuspected wealth of possibilities. It does not seem to be generally realized that offensive and defensive strategies depend strongly upon what each side knows about the other's plans, capabilities and resources. For example, can the offense see the defensive allocation of missiles to targets before allocating weapons to targets? Or must each side allocate in ignorance of the other? Does the offense know that the defense is allocating missiles to individual targets, or defending any of a group of targets with its missile stockpile? Does the defense know which targets have been destroyed, and cease allocating defensive missiles to them? Can the defense predict at which target an offensive weapon is directed, at the time a defensive missile is assigned to that weapon? What does the defense do if he does not know the offensive stockpile size? Questions such as these hint at the almost endless variety of possible models based on different states of knowledge.

It may be prudent for the defense to select an allocation strategy which makes as few assumptions as possible about the offensive stockpile and strategy. Although such a defensive strategy will perform less well than one using more assumptions, when these additional assumptions are true, it may perform much better than one using more assumptions, when these additional assumptions are false. The above is one way of attempting to produce a strategy which performs reasonably well under a variety of conditions. Such strategies are often called robust strategies; for example, attempts to make damage proportional to attack size tend to lead to robust strategies. Robustness is clearly a desirable property of a strategy. Unfortunately, very little investigation has been made of the robustness of the strategies in this monograph. One can only reason from analogy in the field of mathematical statistics, where for many years statisticians have investigated the properties of various estimators of population parameters when assumptions about the nature of the population are incorrect.

Mathematical analysis is frequently easier to carry out if idealized models are postulated. For example, if the area of a target is small with respect to the destructive area of a weapon, it is convenient to treat it as a point target. On the other hand, if the area is large, the target may be approximated by a uniform or a Gaussian distribution of value, instead of an irregular two-dimensional figure. Similarly, in order to avoid integer constraints on missiles or targets, it may be convenient to assume that large numbers of missiles or targets are present and use real numbers instead. Two limiting models of defensive missile performance are especially useful. In one, missiles are assumed to have perfect (or near-perfect) reliability, so that one-on-one engagements are the only ones that occur. In the other, a very large number of individually unreliable missiles are postulated, so that any probability of successful interception can be achieved by a suitable allocation. As another example of useful limiting cases, offensive weapons may be assumed to arrive and be dealt with either one at a time, or all at once: intermediate cases are generally not considered. In general, partially overlapping defensive missile coverage is not considered; instead, one assumes that all defensive missiles can defend all targets in a group, and different groups are defended by independent sets of missiles. It is usually possible to bracket the real situation by one or more of the idealized ones, and from the idealized ones obtain some notion of reasonable defense strategies to use.

1.3 ANOTHER SURVEY OF THE MISSILE ALLOCATION PROBLEM

Matlin (1970) is the only author prior to this monograph to attempt a general survey of the missile allocation problem. He briefly analyzes a total of 40 unclassified articles: ten papers published in the *Journal of the Operations Research Society of America*, one talk given at an ORSA meeting, and twenty-nine government and company reports (Rand Corporation, Lambda Corporation, Analytic Services Operation, Stanford Research Institute, Boeing, General Electric, etc.). Each article is represented by a brief abstract; it is necessary to go back to the original references for the analytic formulas or computing algorithms.

Matlin proposes that all articles relating to the missile allocation problem be fitted into a nine-part classification system:

1. weapon scope (one or more weapon types? decoys?)
2. weapon reach (what payloads can reach which targets?)
3. weapon commitment (one or more waves? launch reliability? offensive damage assessment?)

4. target types (point targets? area targets? collateral damage?)

5. target value (equal values? ranked in value? different values? value of defensive system?)

6. defense level (undefended? terminal defense? area defense also? defense allocation unknown to offense?)

7. engagement model (hard or soft targets? reliable or unreliable defensive missiles?)

8. damage model (zero-one or probabilistic? total or partial damage from a single weapon?)

9. algorithm used to calculate optimum allocation.

Unfortunately, not all missile allocation models fit conveniently into this somewhat Procrustean bed; Matlin creates a special category for models in which a probability p_{ij} is given that the ith weapon penetrates the defense and destroys target j.

Many of the above concepts are treated in this monograph. However, there is a subtle difference between Matlin's work and this monograph — his article is organized around the concept of optimizing offensive weapon allocations, whereas this work is organized around the concept of optimizing defensive missile allocations. (Of course, there is considerable overlap between the two topics, especially when two-sided optimizations must be considered.) Reflecting the latter philosophy, the basic chapter organization of this monograph proceeds from the no-defense case to isolated targets, groups of equal-valued targets, groups of targets with different values, and finally interactions between the targets and their defense system. Chapter 4, the core of the book, considers a large number of defense models principally distinguishable by the degree of knowledge the offense and defense have about each other and about the nature of the engagement; such knowledge is less emphasized by Matlin. On the other hand, Matlin places far more stress on such topics as decoys, multiple weapon types, and weapons reach than this monograph does. Hence, the two surveys should be regarded as complementary rather than competitive.

A second difference between Matlin's work and this monograph is the degree of organization of the material. As noted above, Matlin classifies all papers in a fairly rigid system, but this monograph has adopted the contrasting philosophy of letting the existing papers suggest the overall organization. Because the subject of missile allocation has developed in an uneven fashion, this sometimes leads to unexpected changes in the narrative. To ameliorate this confusion, detailed chapter headings and summaries at the end of each chapter have been introduced.

1.4 A SURVEY OF MATHEMATICAL TECHNIQUES

Even a casual reader will observe that a great variety of mathematical models are to be found in this monograph. It is, therefore, somewhat surprising that most methods of analysis can in fact be loosely assigned to a small number of general classes. Although there is enormous variety within each class, such a classification is useful in studying the relationships between various models and solutions.

The remarks to follow apply primarily to Chapters 3 through 6, and to a much smaller extent to Chapter 2. In fact, the content of Chapter 2 is somewhat anomalous; the characteristic techniques are evaluations of (or approximations to) multiple integrals, and estimation of unknown parameters from data.

Undoubtedly the most prevalent technique to be found is that of permitting the continuous variation of parameters which in reality can take on only integral values. A typical example of this approach can be found in Section 3.1. This technique creates round-off problems which can sometimes make the results useless. However, the round-off problem can often be controlled as demonstrated in Section 4.3.3. The continuous approach may well be selected more often than necessary, because fewer people are trained in the use of discrete methods than continuous ones. Although discrete methods are not as difficult to use as commonly supposed, frequently there is considerable insight to be gained by the continuous approach.

Standard elementary optimization is a technique which often goes hand-in-hand with the above. Although it is based on straightforward calculus and algebra, the analysis can sometimes become rather involved, as demonstrated in Section 3.1. In addition, considerable care must be exercised to include all endpoint optima and multiple stationary points that occur when an objective function assumes different forms in different regions; for example, see Section 5.5.1. The references in this monograph contain many examples of failure to exercise such care; often such errors have been corrected without comment.

The technique of Lagrange multipliers is widely used for finding constrained optima. Often this is just a simple extension of standard elementary optimization, but it can also be applied to problems well outside the context discussed in textbooks. For instance, it can be applied to problems in which the variables are integers; see Section 5.2.2 for a fairly general discussion of the principles involved. The Lagrangian approach can even be applied to max-min problems, as in Section 5.3.1; however, in this case the procedure generally leads to only an approximate solution.

Two very important techniques, sometimes used together, are those of game theory and linear programming. For example, these occur in an indirect form in Sections 4.3.1 - 4.3.6. It is somewhat surprising that game theory has not been explicitly used more often in missile allocation problems, since game theory can be defined as the mathematical theory of conflict. This seems to be in part due to the fact that most such problems are too hard for game theory to be applied directly. It can be argued, however, that the basic concepts of game theory underlie most of the field of missile allocation.

Another frequently-used technique is that of dynamic programming. Where it can be used at all, it tends to be very effective. It can take many forms; a simple introduction to the concept is to be found in Section 5.2.1. It is best applied with the aid of a digital computer.

Another technique sometimes used has no convenient name, although it might be called direct optimization. To apply this technique, one determines a condition that the optimum must satisfy, finds a point that satisfies it, and shows that the point is unique. Although this idea is often involved in other optimization techniques, it can often have the status of an independent technique; Danskin's application of Gibbs' lemma in Section 5.1 is an example. The technique can also be applied to discrete problems, as in Section 4.6.1.

Yet another technique is the Monte Carlo method. Here the problem is simulated probabilistically a large number of times and the results averaged to give an approximation to the expected outcome. An example of its application is to be found at the end of Section 3.6. The method is widely applicable, but it is generally slow and expensive, and its accuracy is often too low for satisfactory sensitivity studies. However, it can be of great value when other approaches fail.

A final technique is the prosaic one of searching among a set of possibilities, as in Section 4.2. It usually appears as a step in the application of other methods, as in Section 5.2.2. When two or more dimensions are involved, the search can become expensive, and rather sophisticated analysis may be needed to make the problem tractable.

The techniques emphasized in this monograph tend to be those that do not require a digital computer to implement. This may seem to be an unnecessary restriction, since almost all users of this monograph are likely to have access to a computer. The two primary reasons for this preference are that computer-oriented solutions are often ad hoc and difficult to describe concisely, and often fall outside the notion of a mathematical model. Because of this, they frequently confer less insight than a more analytical approach. The authors of this monograph are not opposed to computer-oriented techniques. Indeed, the construction of effective, practical algorithms for a computer often is a very challenging problem worthy of serious study. This field is often ignored by both mathematicians and engineers, which is unfortunate, since an efficient algorithm can make the difference between the feasibility or the infeasibility of a method.

1.5 SOME COMMENTS ON TERMINOLOGY AND NOTATION

Each chapter of this monograph is essentially self-contained, and the necessary terminology and notation is either defined as needed or used in a self-explanatory way. Nevertheless, some terminology and notation is widely used with rather consistent meaning in this monograph, and it seems worthwhile to summarize it here.

In the typical conflict situation, one side, called the offense, has a stockpile of A devices, called weapons, with which to attack a target or targets of value to the other side, called the defense. If the targets consist of isolated points, the number of such targets is denoted by T. The defense has a stockpile of D interceptors, called missiles, with which to defend the targets. (However, in Chapter 3, A and D are denoted by n and m, respectively.) On a few occasions, the word weapon is used to mean both offensive weapons and defensive missiles; these occasions will be clear from context. Frequently, it is desirable to consider the stockpiles on a per-target basis, in which case normalized stockpiles $a = A/T$ and $d = D/T$ will be used.

The probability that a missile will destroy the weapon it is assigned to is called its reliability, and is designated by ρ. The probability that an unintercepted weapon will destroy its target is designated by p. The probability that an unintercepted weapon fails to destroy its target is designated by $q_0 = 1 - p$. The probability that a weapon to which a missile has been assigned fails to destroy its target is designated by $q_1 = 1 - p(1-\rho)$. If targets have different values, the various values are designated by $v_1, v_2, \ldots, v_T$. The expected value surviving an engagement is designated E(V); if one is interested in the expected fraction of value surviving, this is designated E(f).

Certain mathematical notation is used throughout this monograph. [x] denotes the greatest integer $\leq x$; brackets are used only for this purpose, never for grouping. The symbol $\binom{n}{k}$ denotes the binomial coefficient $n!/k!\,(n-k)!$; $\Gamma(x)$ denotes the gamma function. Almost all logarithms are natural; to avoid confusion, they will nevertheless be written $\log_e$. In a few places the base of the logarithm makes no difference, and is omitted. The expression Pr( ) means the probability of the event described within the parentheses. The usual convention that various sums and products equal zero and one, respectively, are used. Also, the convention $0^0 = 1$ will be adopted when the expression represents a probability.

1.6 SUMMARY

This chapter sets the stage for the rest of the monograph with a discussion of the various criteria by which different offense-defense strategies are to be judged, some comments on mathematical models, and a list of topics that are (and are not) emphasized. Specifically, this monograph is contrasted with a survey article by S. Matlin in the 1970 Journal of the Operations Research Society of America; the two should be regarded as complementary in that Matlin orients his survey around the offense whereas this survey is oriented toward the defense. The chapter concludes with a summary of mathematical techniques commonly required, as well as terminological and mathematical conventions to be followed.

CHAPTER TWO

POINT AND AREA TARGETS IN THE NO-DEFENSE CASE

At first glance the subject-matter of this chapter may appear to be rather elementary. No questions of offense or defense strategies are involved; one is interested solely in calculating the probability that a point target survives a salvo of one or more weapons. If the target has an extended area, the probability of survival is replaced by the expected fraction of the target surviving. One might reasonably conclude that a few simple mathematical arguments involving independent random events are all that is required.

However, appearances are deceptive. Since World War II a large number of authors have dealt with problems of this type and the results of their researches are widely scattered through the mathematical literature under the general name of coverage problems. A few answers can be obtained in closed form, but the majority run into difficulties which can be overcome only by numerical integration or simulation. This chapter attempts to classify these researches into a more or less logical pattern, emphasizing ideas and results rather than derivations.

This chapter is written for the engineer rather than the mathematician. Specifically, it is restricted to two-dimensional coverage problems rather than n-dimensional ones. Furthermore, little if any attention is given to that part of the literature which deals with the mathematical properties of various probability density functions useful in coverage problems. The reader interested in these details is referred to Ruben (1960). Part of the material in this chapter is discussed in two excellent review articles on coverage problems by Guenther and Terragno (1964) and Guenther (1966).

The results of this chapter may be useful for calculating defensive intercept probabilities as well as offensive success probabilities. Specifically, a point "target" can be identified as an incoming attack weapon, and the "weapon" can be identified as a defensive missile. Of course, this transformed problem is three-dimensional; however, it may sometimes be possible to use a two-dimensional approximation if the defensive missile aiming error is small in one dimension.

Much of the material in Sections 2.1 through 2.6 in this chapter was earlier published as a survey article on coverage problems in Eckler (1969). However, the reader should be warned that in certain sections substantial revisions have been made. A very useful general reference, containing properties, tables, etc., of many of the special functions appearing in this chapter is Abramowitz and Stegun (1964).

2.1 SURVIVAL/DESTRUCTION PROBABILITIES FOR ONE OR MORE POINT TARGETS

When the size of the target is small compared with the effective radius of action of the weapon, it is represented by a point in the mathematical model of the attack. As will be shown below, point targets in general are relatively simple to deal with. In particular, salvos of weapons can be handled with ease: if P denotes the probability that a single weapon destroys the target, then the probability that it is destroyed by at least one out of n independently aimed identical weapons is given by $1 - (1-P)^n$. This assumes, of course, that damage is not cumulative: that is, if one weapon fails to destroy the target, the probability that another one does is unchanged. In particular, area targets do not have this simple property except in certain trivial situations.

Assume that a point target is located at the origin of coordinates (0,0) in the xy-plane. Denote the probability density function of the impact-point of the weapon by p(x,y), and let the probability of target destruction be given by the damage function d(x,y) if the weapon impacts at (x,y). Then the unconditional probability of target destruction by a single weapon is given by

$$P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(x,y)\, p(x,y)\, dx\, dy \quad .$$

One can note in passing that this formula is also useful if the target can incur partial damage; in this case, d(x,y) may be taken to represent the expected fraction of the target destroyed if a weapon impacts at (x,y), and P becomes the unconditional expected fraction destroyed.

In general, the damage function is circularly symmetric — that is, the probability of destruction is a function of the single variable $r = \left(x^2 + y^2\right)^{1/2}$. Furthermore, it is a nonincreasing function of r. In this section, a damage function is assumed in which d(x,y) equals unity when $r \leq R$ and zero elsewhere. Hereafter, this will be called a cookie-cutter damage function. One then has

$$P = \iint\limits_{\left(x^2+y^2\right)^{1/2} \leq R} p(x,y)\, dx\, dy \quad .$$

The remainder of this section is concerned with the problem of evaluating P for various choices of the impact-point probability density function p(x,y). In general, p(x,y) is assumed to have a bivariate Gaussian probability density function:

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{\left(x-x_0\right)^2}{2\sigma_x^2} - \frac{\left(y-y_0\right)^2}{2\sigma_y^2}\right),$$

where $\left(x_0,y_0\right)$ is the mean of the impact-point distribution, σ_x^2 is the variance in the x-coordinate, and σ_y^2 the variance in the y-coordinate. Note that the covariance σ_{xy} is assumed to be zero; this can always be achieved by proper choice of the x- and y-axes. Four cases are considered:

(a) $x_0 = y_0 = 0;\ \sigma_x^2 = \sigma_y^2\left(= \sigma^2\right)$.

(b) $x_0 \neq 0,\ y_0 \neq 0;\ \sigma_x^2 = \sigma_y^2\left(= \sigma^2\right)$.

(c) $x_0 = y_0 = 0;\ \sigma_x^2 \neq \sigma_y^2$.

(d) $x_0 \neq 0,\ y_0 \neq 0;\ \sigma_x^2 \neq \sigma_y^2$.

2.1.1 Equal Variances, Distribution Centered at Origin

If $\sigma_x^2 = \sigma_y^2 = \sigma^2$ and $x_0 = y_0 = 0$, then p(x,y) is a function of r alone and can be rewritten

$$p(r) = \left(r/\sigma^2\right) \exp\left(-r^2/2\sigma^2\right) .$$

This is known as a Rayleigh distribution, and is frequently used in problems of noise theory. It is closely related to the exponential distribution (the chi-squared distribution with two degrees of freedom), which is the probability density function of $x^2 + y^2 = r^2$:

$$f\left(r^2\right) = \left(1/2\sigma^2\right) \exp\left(-r^2/2\sigma^2\right) .$$

It is easy to write down the probability of target destruction in closed form:

$$P(R/\sigma) = \int_0^R \left(r/\sigma^2\right) \exp\left(-r^2/2\sigma^2\right) dr = 1 - \exp\left(-R^2/2\sigma^2\right) .$$

2.1.2 Equal Variances, Offset Distribution

If $\sigma_x^2 = \sigma_y^2 = \sigma^2$ but $x_0 \neq 0$, $y_0 \neq 0$, the destruction probability P is a function of the offset aiming-point distance $r_0 = \left(x_0^2 + y_0^2\right)^{1/2}$. Rotating the coordinates axes so that r_0 lies along the positive x-axis,

$$P\left(R/\sigma, r_0/\sigma\right) = \iint_{\left(x^2+y^2\right)^{1/2} \leq R} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{\left(x-r_0\right)^2}{2\sigma^2} - \frac{y^2}{2\sigma^2}\right) dx\, dy \quad .$$

Transforming to polar coordinates by setting $x = r\cos\theta$, $y = r\sin\theta$, this becomes

$$P\left(R/\sigma, r_0/\sigma\right) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r_0^2}{2\sigma^2}\right)$$

$$\cdot \int_0^R \int_0^{2\pi} r \exp\left(-\frac{r^2}{2\sigma^2} + \frac{r r_0 \cos\theta}{\sigma^2}\right) d\theta\, dr \quad .$$

Integrating out the variable θ,

$$P\left(R/\sigma, r_0/\sigma\right) = \left(1/\sigma^2\right) \exp\left(-r_0^2/2\sigma^2\right)$$

$$\cdot \int_0^R r \exp\left(-r^2/2\sigma^2\right) I_0\left(r r_0/\sigma^2\right) dr \quad ,$$

where $I_n(z) = J_n(iz)\exp(-\pi n i/2)$ is a modified Bessel function of the first kind of order n.

This function cannot be integrated in closed form, but several tables or graphs of $P\left(R/\sigma, r_0/\sigma\right)$ are available:

Bell Aircraft Corporation (1956): $P\left(R/\sigma, r_0/\sigma\right)$ to 5 decimals for $r_0/\sigma = 0(0.01)3$,
$R/\sigma = 0.01(0.01)4.59$.

Marcum (1950): $1 - P\left(R/\sigma, r_0/\sigma\right)$ to 6 decimals for
$R/\sigma = 0.1(0.1)20$, r_0/σ = by intervals of 0.05
to cover a range of P from 0 to 1 .

Rand Corporation (1952); also Owen (1962):
$1 - P\left(R/\sigma, r_0/\sigma\right)$ to 3 decimals for
$\left(R-r_0\right)/\sigma = -3.9(0.1)4.0$,
$r_0/\sigma = 0.1(0.1)6(0.5)10(1)20$.

Burington and May (1953): $P\left(R/\sigma, r_0/\sigma\right)$ to 3 decimals
(4 decimals for $P < .01$) for $R/\sigma = 0(0.1)1(0.2)3$,
$r_0/\sigma = 0(0.1)3(0.2)6$.

DiDonato and Jarnagin (1962): R/σ to 7 significant figures for $r_0/\sigma = 0(0.1)5(0.2)10(2)20(5)120$,
$P = 0.01(0.01)0.99$.

Solomon (1953): Figure 1 depicts $P = 0.05(0.05)0.95$ graphed over ranges $0 \le r_0/\sigma \le 10$ (horizontal axis) and $0 \le R/\sigma \le 8$ (vertical axis) .

Rice (1945): Figure 7 depicts $r_0/\sigma = 0, 1, 2, 3, 5, \infty$ graphed over ranges $0.0001 \le P \le 0.9999$ (vertical axis) and $-4 \le \left(R-r_0\right)/\sigma \le 4$ (horizontal axis) .

Groves and Smith (1957): Figure 2 depicts
$R/\sigma = 0.1, 0.5, 1(1)12$ graphed over ranges
$0 \le r_0/\sigma \le 9$ (horizontal axis) and
$.0001 \le 1 - P \le .9999$ (vertical axis) .

If none of these tables or graphs is available, many approximations to $P\left(R/\sigma, r_0/\sigma\right)$ have been proposed which use functions commonly available in statistical tables. Gilliland (1962) suggests approximating $P\left(R/\sigma, r_0/\sigma\right)$ by the first few terms of the infinite series

$$P\left(R/\sigma, r_0/\sigma\right) = \exp\left(-r_0^2/2\sigma^2\right) \sum_{m=0}^{\infty} \left(r_0^2/2\sigma^2\right)^m P_{m+1}\left(R^2/2\sigma^2\right) \Big/ m!$$

$$= \sum_{m=0}^{\infty} \left(P_m\left(r_0^2/2\sigma^2\right) - P_{m-1}\left(r_0^2/2\sigma^2\right)\right) P_{m+1}\left(R^2/2\sigma^2\right) ,$$

where $P_m(\lambda) = \sum_{u=m+1}^{\infty} e^{-\lambda}\lambda^u/u!$, the upper tail of the Poisson distribution. The more complex second expression may be useful for hand computation, since it uses values from a single table. Grubbs (1964a) suggests approximating $P(R/\sigma, r_0/\sigma)$ by looking up in statistical tables the probability that a chi-squared random variable with $[2t^2/v]$ degrees of freedom is less than or equal to $tR^2/v\sigma^2$, where $t = 1 + r_0^2/2\sigma^2$ and $v = 1 + r_0^2/\sigma^2$. Read (1971) cites a Rand Corporation memorandum which suggests three different approximations for $P(R/\sigma, r_0/\sigma)$, depending upon the values of R/σ and r_0/σ. Two of the approximations use exponential functions, and the third uses the cumulative distribution function of a Gaussian variable. It is claimed that the maximum error is 0.02. A similar (but more accessible) approximation to $P(R/\sigma, r_0/\sigma)$ is given on p. 940 of Abramowitz and Stegun (1964):

$$(1) \quad P(R/\sigma, r_0/\sigma) \simeq \frac{2R^2}{4+R^2} \exp\left(-\frac{-2r_0^2}{4+R^2}\right) \quad \text{if} \quad R < 1 ;$$

$$(2) \quad P(R/\sigma, r_0/\sigma) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_1} \exp\left(-x^2/2\right) dx$$

$$= \frac{1}{2}\left(1 + \operatorname{erf}\left(x_1/\sqrt{2}\right)\right) \quad \text{if} \quad R > 1 ,$$

$$\text{where} \quad x_1 = \frac{\left(R^2/(2+r_0)\right)^{1/3} - 1 + (2/9)\left(2+2r_0^2\right)/\left(2+r_0^2\right)^2}{\left((2/9)\left(2+2r_0^2\right)/\left(2+r_0^2\right)^2\right)^{1/2}} ;$$

$$(3) \quad P(R/\sigma, r_0/\sigma) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_2} \exp\left(-x^2/2\right) dx$$

$$= \frac{1}{2}\left(1 + \operatorname{erf}\left(x_2/\sqrt{2}\right)\right) \quad \text{if} \quad R > 5 ,$$

$$\text{where} \quad x_2 = R - \left(r_0^2 - 1\right)^{1/2} .$$

Finally, Brennan and Reed (1965) suggest a recursive method of computing the function $P\left(R/\sigma, r_0/\sigma\right)$ with the aid of a digital computer to any desired degree of accuracy. Specifically, they replace the Bessel function $I_0(x)$ by its series expansion

$$\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^{2n}\frac{1}{(n!)^2} .$$

$P\left(R/\sigma, r_0/\sigma\right)$ can be written in the form

$$P\left(R/\sigma, r_0/\sigma\right) = 1 - \sum_{i=1}^{\infty} g_n k_n ,$$

where

$$g_n = \frac{1}{n!}\left(\frac{R^2}{2\sigma^2}\right)^{n+1}\int_0^1 u^n \exp\left(-R^2 u/2\sigma^2\right) du$$

$$= g_{n-1} - \frac{1}{n!}\left(\frac{R^2}{2\sigma^2}\right)^n \exp\left(-R^2/2\sigma^2\right) ,$$

$$k_n = \left(\frac{r_0^2}{2\sigma^2}\right)^n \frac{1}{n!}\exp\left(-r_0^2/2\sigma^2\right) = \frac{r_0^2}{2\sigma^2}\cdot\frac{k_{n-1}}{n} .$$

Details of the error involved in truncating this series are given in the original paper.

2.1.3 Unequal Variances, Distribution Centered at Origin

Suppose now that $x_0 = y_0 = 0$ but $\sigma_x^2 \neq \sigma_y^2$. Without loss of generality, one may assume $\sigma_x \geq \sigma_y$; denote σ_x by σ_{max}. Let c be the ratio of the smaller variance to the larger; that is, $c = \sigma_y^2/\sigma_x^2$. Transforming to polar coordinates by setting $x/\sigma_{max} = r\cos\theta$, $y/\sigma_{max} = r\sin\theta$, the probability of destruction becomes

$$P\left(R/\sigma_{max}, c\right) = (1/2\pi c) \int_0^{R/\sigma_{max}} \int_0^{2\pi} r$$

$$\cdot \exp\left(-\left(r^2/2\right)\left(\cos^2\theta + \left(\sin^2\theta/c^2\right)\right)\right) d\theta\, dr \;.$$

Transforming to the variable $\phi = \theta/2$, one obtains

$$P\left(R/\sigma_{max}, c\right) = (1/2\pi c) \int_0^{R/\sigma_{max}} \int_0^{\pi} r$$

$$\cdot \exp\left(-\left(r^2/4c^2\right)\left(\left(1+c^2\right) - \left(1-c^2\right)\cos\phi\right)\right) d\phi\, dr$$

$$= (1/c) \int_0^{R/\sigma_{max}} r$$

$$\cdot \exp\left(-r^2\left(1+c^2\right)/4c^2\right) I_0\left(r^2\left(1-c^2\right)/4c^2\right) dr \;.$$

Harter (1960) derives an alternative form of $P\left(R/\sigma_{max}, c\right)$ more suitable for numerical integration. Let $z = r^2/4c^2$ in the second equation above, and integrate with respect to z:

$$P\left(R/\sigma_{max}, c\right) = \frac{2c}{\pi} \int_0^{\pi} \left(\left(1+c^2\right) - \left(1-c^2\right)\cos\phi\right)^{-1}$$

$$\cdot \left(1 - \exp\left\{-\left(r^2/4c^2\right)\left(\left(1+c^2\right) - \left(1-c^2\right)\cos\phi\right)\right\}\right) d\phi$$

The function $P\left(R/\sigma_{max}, c\right)$ cannot be integrated in closed form, but is tabulated in several places:

Solomon (1960): $P\left(\left(a_2, a_1\right), t\right)$ to 8 decimals for $\left(a_2, a_1\right)$ = (.5,.5),(.6,.4),(2/3,1/3),(.7,.3), (.75,.25),(.8,.2),(.875,.125),(.9,.1), (.95,.05),(.99,.01) and t = .005(.005).1(.01) 1(.02)2.5(.025)3.5(.05)5(.25)6(.5)7(1)10 .

To transform to the above variables, set $c^2 = a_1/a_2$ and $(R/\sigma_{max})^2 = t/a_2$.

Grad and Solomon (1955): $P((a_2,a_1),t)$ to 4 decimals for $(a_2,a_1) = (.5,.5),(.6,.4),(.7,.3),(.8,.2),(.9,.1),(.95,.05),(.99,.01)$ and $t = .1(.1)1(.5)2(1)5$.

Esperti (1960): $P(R/\sigma_{max},c)$ to 6 decimals for $R/\sigma_{max} = 0.00(0.01)4.99$, $c = 0(0.1)1.0$.

Beyer (1966): $P(c,R/\sigma_{max})$ to 7 decimals for $R/\sigma_{max} = 0.1(0.1)5.5$, $c = 0(0.1)1.0$.

Harter (1960): $P(R/\sigma_{max},c)$ to 7 decimals for $R/\sigma_{max} = 0.1(0.1)6$, $c = 0(0.1)1.0$, and R/σ_{max} to 6 significant figures for $c = 0(.1)1$, $P = .5,.75,.9,.95,.975,.99,.995,.9975,.999$.

DiDonato and Jarnagin (1962): R/σ_{max} to 7 significant figures for $c = 0,.1(.05)1$, $P = .99(0005).999(.0001).9999(.00001).99999(.000001).999999$.

Weingarten and DiDonato (1961): R/σ_{max} to 6 significant figures for $c = .05(.05)1$, $P = .05(.05).95(.01).99$.

If σ_y and σ_x are not too greatly different (say, $c \geq 0.5$), it is tempting to calculate an approximate radius, R, for the circle which contains P of the bivariate probability. Oberg (1947) suggests three such approximations. He notes first that the ellipse centered on (0,0) with semi-axes $\sigma_x\left(2\log_e(1/(1-P))\right)^{1/2}$ and $\sigma_y\left(2\log_e(1/(1-P))\right)^{1/2}$ contains exactly P of the bivariate probability, and suggests using the radius of the circle with the same area. This gives

$$R_1 = \left(2\sigma_x\sigma_y \log_e\left(1/(1-P)\right)\right)^{1/2} .$$

If $\sigma_x = \sigma_y$, it has already been proved that $P = 1 - \exp\left(-R^2/2\sigma^2\right)$. If $2\sigma^2$ is replaced with $\sigma_x^2 + \sigma_y^2$, a second approximate R is obtained:

$$R_2 = \left(\sigma_x^2 + \sigma_y^2\right)\left(\log_e \left(1/(1-P)\right)\right)^{1/2} .$$

Finally, one can combine R_1 and R_2 into a third estimate:

$$R_3 = \left(\left(R_1^2 + R_2^2\right)/2\right)^{1/2} .$$

Oberg gives tables showing the relative performance of R_1, R_2 and R_3 for $0.1 \leq P \leq 0.9$ and $0.5 \leq c \leq 1$. However, much better approximations (not dependent upon c being near unity) are available. Gilliland (1962) suggests approximating $P\left(R/\sigma_{max}, c\right)$ by the first few terms of the infinite series

$$P\left(R/\sigma_{max}, c\right) = \frac{2c}{1+c^2} \sum_{m=0}^{\infty} \binom{2m}{m} 2^{-2m}$$

$$\cdot \left(\frac{1-c^2}{1+c^2}\right)^{2m} P_{2m}\left(\left(1+c^2\right) R^2/4c^2\right) ,$$

where, as before, $P_m(\lambda) = \sum_{u=m+1}^{\infty} e^{-\lambda} \lambda^u/u!$, the upper tail of the Poisson distribution. Grubbs (1964a) approximates $P\left(R/\sigma_{max}, c\right)$ by means of the probability that a Gaussian distribution of zero mean and unit variance is less than

$$\frac{3}{\sqrt{2}} \cdot \frac{\left(R^2\left(\sigma_x^2 + \sigma_y^2\right)\right)^{1/3} - 1 + 2\left(\sigma_x^4 + \sigma_y^4\right)/9\left(\sigma_x^2 + \sigma_y^2\right)^2}{\left(\sigma_x^4 + \sigma_y^4\right)^{1/2}/\left(\sigma_x^2 + \sigma_y^2\right)} .$$

This is the Wilson-Hilferty transformation of a chi-squared random variable to a Gaussian random variable, and is given in Equation (20) in the paper by Grubbs. Lilliefors (1957) also approximates $P\left(R/\sigma_{max}, c\right)$ by the first few terms of an infinite series. Note that his approximation requires no tabulated functions whatever:

$$P\left(R/\sigma_{max}, c\right) = \frac{\sigma_y}{\sigma_x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \left(R^2/2\sigma_y^2\right)^n a_1(n) ,$$

where $a_n(n) = 1$,

$$a_j(n) = Aa_{j+1}(n) + Bb_{j+1}(n), \qquad 1 \le j \le n-1\ ,$$

$$b_n(n) = 0,$$

$$b_j(n) = \Big((j-1)/j\Big)\Big(Ba_{j+1}(n) + Ab_{j+1}(n)\Big)\ , \qquad 2 \le j \le n-1\ ,$$

and where

$$A = 1 + \left(R^2/4\sigma_x^2\right) \quad \text{and} \quad B = \left(\left(\sigma_y^2/\sigma_x^2\right) - 1\right)/2\ .$$

The first few $a_1(n)$ are:

$$a_1(1) = 1,$$

$$a_1(2) = A,$$

$$a_1(3) = A^2 + B^2/2,$$

$$a_1(4) = A^3 + 3B^2A/2,$$

$$a_1(5) = A^4 + 3B^2A^2 + 3B^4/8\ .$$

However, higher $a_1(n)$ become increasingly tedious to compute.

2.1.4 Unequal Variances, Offset Distribution

If $x_0 \neq 0$, $y_0 \neq 0$ and $\sigma_x^2 \neq \sigma_y^2$, the destruction probability P is a function of four variables which can be selected in various (equivalent) ways. Gilliland (1962) has derived the following expression for P:

$$P = \frac{1}{2\pi c} \exp\left\{-\frac{1}{2}\left(\frac{x_0^2}{\sigma_x^2} + \frac{y_0^2}{\sigma_y^2}\right)\right\} \sum_{m=0}^{\infty} B_m P_m\left(\left(1+c^2\right) R^2/4c^2\right)\ ,$$

where c and $P_m(\lambda)$ are defined as before, and

$$B_m = \frac{1}{2}\, m!\, \left(4c^2/\left(1+c^2\right)\right)^{m+1} \sum_{i=0}^{m} D_{mi}\ ,$$

where

$$D_{mi} = \left(\frac{1-c^2}{4c^2}\right)^i \frac{1}{i!} \sum_{j=0}^{m-i} \frac{\left(x_0^2/\sigma_x^2\right)^j \left(y_0^2/c^2\sigma_y^2\right)^{m-i-j}}{(2j)!\,(2m-2i-2j)!}$$

$$G(i,2j,2m-2i-2j) \quad ,$$

with

$$G(p,q,t) = \int_0^{2\pi} \cos^p 2\theta \cos^q\theta \sin^t\theta \, d\theta \quad .$$

Again, several tables are available.

DiDonato and Jarnagin (1960): R is given to 5 significant figures for P = .05,.2,.5,.7, .9,.95, x_0 = 0(0.25)1(1)6,8,10,20,50, y_0 = 0(0.5)1(1)6(2)10(10)30,50,80,120, and 1/c = 1(1)8,10 .

Groenewoud, Hoaglin and Vitalis (1967): P is given to 5 decimals for x_0/σ_x = 0(0.5)5, y_0/σ_y = 0(0.5)5, $\sqrt{c}$ = 0.1(0.1)1, and R/σ_{max} = 0 for those values for which P is nearly one.

Rosenthal and Rodden (1961): P is given to 5 decimals for c = 0.2(0.2)1, x_0/σ_x = 0(0.5)3, y_0/σ_y = 0(0.5)3, and R/σ_{max} = 0 for those values for which P is nearly one, in steps of .05.

Lowe (1960): P is given to 3 decimals. For various choices of 1/c and R/σ_y, tables of 64 entries of P are presented in which 8 values of x_0/σ_x and 8 values of y_0/σ_y are selected to cover the region in which the variation in P is applicable. The following 36 tables are given:

1/c	R/σ_y	Range of x_0/σ_x	Range of y_0/σ_y
1	1	0 to 3.01	0 to 3.01
1	2	0 to 4.06	0 to 4.06
1	4	0 to 6.02	0 to 6.02
1	8	4.5 to 10.03	0 to 2.03
1	8	0 to 10.01	2 to 5.01
1	8	0 to 8.68	5 to 10.04

$1/c$	R/σ_y	Range of x_0/σ_x	Range of y_0/σ_y
2	1	0 to 2.52	0 to 3.01
2	2	0 to 3.01	0 to 4.06
2	4	0 to 4.06	0 to 6.02
2	8	0 to 6.02	0 to 4.55
2	8	0 to 5.46	4.5 to 10.02
2	16	4 to 10.02	0 to 7.00
2	16	0.75 to 9.29	7 to 12.81
2	16	0 to 7.28	12.75 to 18.00
4	1	0 to 2.31	0 to 3.01
4	2	0 to 2.52	0 to 4.06
4	4	0 to 3.01	0 to 6.02
4	8	0 to 4.06	0 to 10.01
4	16	0 to 6.02	0 to 10.01
4	16	0 to 5.18	10 to 18.05
4	32	4 to 10.02	0 to 15.05
4	32	2.15 to 9.15	15 to 24.03
4	32	0 to 7.42	24 to 28.55
4	32	0 to 5.88	28.5 to 34.03
8	1	0 to 2.17	0 to 3.01
8	2	0 to 2.31	0 to 4.06
8	4	0 to 2.52	0 to 6.02
8	8	0 to 3.01	0 to 10.01
8	16	0 to 4.06	0 to 18.06
8	32	0 to 6.02	0 to 20.79
8	32	0 to 5.11	20.75 to 34.05
8	64	4 to 10.02	0 to 30.03
8	64	2.9 to 9.13	30 to 42.53
8	64	1.5 to 8.01	42.5 to 52.51
8	64	0 to 6.58	52.5 to 58.8
8	64	0 to 5.25	58.75 to 66.03

Grubbs (1964a) gives the only reasonably simple hand-calculation for P. He shows that P is approximately equal to the probability that a Gaussian distribution of zero mean and unit variance is less than

$$\frac{\left(R^2/\left(\sigma_x^2+\sigma_y^2\right)t\right)^{1/3}-\left(1-v/9t^2\right)}{\left(v/9t^2\right)^{1/2}},$$

where $t = 1 + \left(x_0^2 + y_0^2\right)/\left(\sigma_x^2 + \sigma_y^2\right)$ and $v = 2\left(\sigma_x^4 + \sigma_y^4 + 2\sigma_x^2 x_0^2 + 2\sigma_y^2 y_0^2\right)/\left(\sigma_x^2 + \sigma_y^2\right)^2$. Note that this generalizes the formula of Grubbs given in the previous section.

2.1.5 Targets With More Than One Point

If the target consists of a single point, it is obvious that the weapon should be aimed at it in order to maximize the probability of target destruction — in other words, x_0 and y_0 should be chosen equal to zero if possible. Suppose, however, that the target consists of more than one point; where should one aim a single weapon in order to maximize the probability that it lands within a distance R of at least one of the points?

Consider the simplest possible configuration: two point targets located at $(-d,0)$ and $(d,0)$. The probability density function of the impact point of the weapon is assumed to be circular Gaussian with $\sigma_x^2 = \sigma_y^2 = \sigma^2$. Gilliland (1964) proved that if $R \le d \le \sigma$, then the optimum aim point is (0,0). Marsaglia (1965) derived the general solution to the problem. He provides a graph dividing $(R/\sigma, d/\sigma)$ space into two regions — in one, the optimum aim point is (0,0), and in the other the optimum aim points are $\pm(z,0)$, where z/σ is the positive solution to the equation

$$\int_{-1}^{\min(1,d/R)} \left(1-y^2\right)^{-1/2} y \sinh\left((z/\sigma)(Ry+d)/\sigma\right) \cdot \exp\left(-dRy/\sigma^2\right) dy = 0 \quad .$$

If the target consists of more than two points, the problem of finding the optimum aim point becomes much more difficult. Gilliland (unpublished work) has proved that if three target points are located at the vertices of an equilateral triangle with mutual separation d, and if $R \le d \le 4\sigma/(1+\sqrt{3})$, then the optimum aim point is at the center of the triangle. This is a somewhat stronger result than the one given in Gilliland (1966), which requires $R \le d \le \sigma$.

In the latter reference, he also proved that if a target consists of a symmetric array of points around the origin $\left(x_i, y_i\right)$, $\left(-x_i, -y_i\right)$, $i = 1,2,\ldots,n$, and if these points have the properties $R \le \left(x_i^2 + y_i^2\right)^{1/2} \le \sigma$ (for all i) and $\left(\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2\right)^{1/2} \ge 2R$ (for all $i \ne j$), then the optimum aim point is at the origin. The latter property can be eliminated if the first property is replaced by $\left(x_i + y_i^2\right)^{1/2} \le \sigma - R$. Gilliland (1968) has derived analogous results for a class of bell-shaped probability density functions of the impact-point of the weapon.

2.1.6 Models of Aiming Error Associated With a Salvo

In Sections 2.1.1 through 2.1.4, various models of an attack upon an undefended point target were introduced. These models explicitly formulated the widely-held belief that it is reasonable to regard weapon aiming-error as the sum of two components:

1. a constant offset in aim, equal for all weapons in the salvo,
2. a random error taken from a Gaussian probability density function with standard deviations σ_x and σ_y, and independent from one weapon to the next.

Furthermore, it was assumed that the constant offset was known beforehand. This assumption is rather unsatisfactory (if the offset is known, why not correct for it?), and in the models to be presented later in this chapter, it is usually assumed that the constant bias is a random value drawn from another Gaussian probability density function with known mean and variance.

Helgert (1971) points out that a more realistic model of weapon aiming-error should include not only the above two components but also a time-varying component which he calls aim-wander. This reflects the physical fact that the path traced by the intersection of the gun barrel mean line of sight and a plane perpendicular to it would, as a function of time, appear to wander in a more or less random fashion. In principle, this wander can be mathematically modeled by a sequence of bivariate random variables that are partially correlated with each other (i.e., neither perfectly correlated with each other, as in (1), nor completely uncorrelated with each other, as in (2)). However, it may be difficult to specify the precise correlation in any practical problem; even more important, the mathematical complexities introduced by such a model make it impossible to calculate such quantities as the expected fraction of an area target destroyed by a salvo of n weapons. Helgert, in fact, finds it necessary to calculate a much more elementary measure of target damage: the probability that a specified subset of the n weapons will impact within the target area, given the partial correlations. Perhaps the most interesting of his partial correlation models is a Markov model — one in which the probability that the ith round hits the target depends only upon the success or failure of the (i-1)st round. It is an open question as to whether the Markov model is superior to the two-component model of weapon aiming-error discussed above; this is a matter for further research.

2.2 EXPECTED FRACTIONAL DAMAGE OF A UNIFORM-VALUED CIRCULAR TARGET

As in Section 2.1, only cookie-cutter damage functions will be considered. If the radius of the damage function is large with respect to the size of the target, the latter can be approximated by a point. However, this is not always the case — often, only a fraction of the target is destroyed when a weapon hits it. Assume a target which has uniform value anywhere within a distance K of the origin (0,0) in the xy-plane, and zero value outside K. Let the probability density function of the impact-point of the weapon be $p(x,y)$ and let a point (x_t, y_t) in the target be destroyed with probability $d(x-x_t, y-y_t)$ by a weapon impacting at (x,y). By analogy with the generalization mentioned in Section 2.1, $d(x-x_t, y-y_t)$ may be taken to be the expected fractional damage at the point (x_t, y_t). Then the expected fraction of the target destroyed by a single weapon is equal to

$$E(f) = \iint_{\left(x_t^2+y_t^2\right)^{1/2} \le K} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\left(x-x_t, y-y_t\right) \cdot p(x,y)\, dx\, dy\, dx_t\, dy_t \ .$$

If a cookie-cutter damage function is assumed, this simplifies to

$$E(f) = \iint_A \iint_B p(x,y)\, dx\, dy\, dx_t\, dy_t \ ,$$

where A is the region $\left(x_t^2 + y_t^2\right)^{1/2} \le K$ and B is the region $\left(\left(x-x_t\right)^2 + \left(y-y_t\right)^2\right)^{1/2} \le R$. As before, $p(x,y)$ is ordinarily assumed to have a bivariate Gaussian probability density function. Because of mathematical difficulties, the one generally considered in this section is

$$p(x,y) = \left(1/2\pi\sigma^2\right) \exp\left(-\left(x^2+y^2\right)/2\sigma^2\right) \ ,$$

which is circularly symmetric and centered on the target.

Although the material in this section is discussed in terms of a uniform-valued circular target, it can be equally well applied to a point target having an unknown location. Specifically, assume that the location of the point target is given by a probability density function assigning uniform probability inside the circle $\left(x_t^2 + y_t^2\right)^{1/2} \le K$ and zero outside it. The uncertain location of the point target might be due to a map error, or the target might be mobile, having last been spotted some time before the attack. In the latter case, the radius K is equal to the maximum distance the target could have covered in the elapsed time. (Note that a knowledge of target direction might be helpful in reducing the unknown area.)

2.2.1 One Weapon Impact, Gaussian Aiming Error

The expected fraction of the target covered is a function of R/σ and K/σ. Guenther (1964) shows that it can be written

$$E(f) = \left(1 + (R/K)^2\right) P(R/\sigma, K/\sigma)$$

$$- (R/K) \exp\left(-\left(R^2+K^2\right)/2\sigma^2\right) I_1(RK/\sigma) ,$$

where

$$P(R/\sigma, K/\sigma) = \left(1/\sigma^2\right) \exp\left(-K^2/2\sigma^2\right)$$

$$\cdot \int_0^R r \exp\left(-r^2/2\sigma^2\right) I_0\left(rK/\sigma^2\right) dr .$$

($I_n(z) = J_n(iz) \exp(-\pi ni/2)$ is a modified Bessel function of the first kind of order n.) $P(R/\sigma, K/\sigma)$ is the probability of destruction of a point target by a weapon with a circular Gaussian distribution and offset aiming-point distance K. In other words, the tables of $P\left(R/\sigma, r_0/\sigma\right)$ described earlier can be used to evaluate E(f). One table gives E(f) directly:

Germond (1950): E(f) to 4 decimals for
$K/\sigma = 0(0.1)6.5$, $R/\sigma = 0(0.5)3(1)6$ and for
$(K-R)/\sigma = -3.2(0.1)3.5$, $R/\sigma = 3(1)6(2)20$.

Smith and Stone (1961) derived E(f) under the modified assumption that the center of p(x,y), the weapon impact-point probability density function, is offset a distance r_0 from the center of the target:

$$E_{offset}(f) = 2\pi KR \int_0^\infty (1/r) \exp\left(-r^2/2\right)$$

$$\cdot J_0\left(r_0 r/\sigma\right) J_1(Kr/\sigma) J_2(Rr/\sigma) dr ,$$

where $J_n(z)$ is a Bessel function of the first kind of order n.

If one is interested in the probability P_c that the fraction of the target destroyed exceeds c (instead of E(f), the expected value of this fraction), one can use a pair of graphs supplied by Solomon (1953). Actually, Solomon considers a generalization of the above problem; he allows the mean of the impact-point probability density function, p(x,y), to be circular Gaussian offset a distance r_0 from the center of the target. His graphs yield P_c as a function of R/σ, K/σ and r_0/σ. E(f) can be approximated by reading off the value of c corresponding to $P_c = 0.5$ (the median) in Solomon's graphs. To accomplish this, let $a = \left(1.2 - (K/\sigma)\right)/(R/\sigma)$, enter with this value on the abscissa of Solomon's Figure 2, and read off the appropriate c (a function of a and K/R) on the ordinate.

2.2.2 Multiple Weapon Impacts, Gaussian Aiming Error

What is the expected fraction destroyed of the target if there are n weapons directed at it? Guenther (1966) assumes that each weapon has an impact-point error which is the sum of two independent components: an offset aiming-point error (the same for all weapons), and a dispersion error around the offset aiming-point (different for each weapon). Such a complex model is not absolutely necessary; the offset aiming-point error can be lumped in with the point target location error. In other words, the impact-point errors can be regarded as independent samples from a common probability density function p(x,y). However, since independence does not hold,

$$E_n(f) \neq 1 - \left(1 - E_1(f)\right)^n .$$

In general, $E_n(f)$ will be smaller than the expression on the right. The following derivation of $E_n(f)$ is due to Jarnagin (1966). Let $P(R/\sigma, r/\sigma)$ denote the probability that a point r distant from the origin is destroyed by a weapon; in Section 2.1.2 this was seen to be

$$P(R/\sigma, r/\sigma) = \frac{1}{\sigma^2} \exp\left(-r^2/2\sigma^2\right) \cdot \int_0^R x \exp\left(-x^2/2\sigma^2\right) I_0\left(xr/\sigma^2\right) dx .$$

The probability that this point is destroyed by one or more of n independent weapons is $1 - \left(1 - P(R/\sigma, r/\sigma)\right)^n$. To determine the expected fraction of the target destroyed, this expression must be integrated over the target area: $\left(x_t^2 + y_t^2\right)^{1/2} = r \leq K$.

$$E_n(f) = \left(1/\pi K^2\right) \int_0^{2\pi} \int_0^K \left(1 - P(R/\sigma, r/\sigma)\right)^n r \, dr \, d\theta$$

$$= 1 - \left(2/K^2\right) \int_0^K \left(1 - P(R/\sigma, r/\sigma)\right)^n r \, dr \; .$$

$E_n(f)$ has been tabulated by Jarnagin (1965) as a function of $R/\sigma = 0.005(0.005)0.05(0.01)0.1(0.02)0.2(0.05)1(0.1)2(0.2)4(0.5)10$, $K/\sigma = 0.05, 0.1(0.1)4(0.5)12$, and $n = 1(1)20$. The value of n (when less than 1000) has been tabulated for the same ranges of R/σ and K/σ, and corresponding to $E_n(f) = 0.05(0.05)0.95$.

In a related paper, Jarnagin and DiDonato (1966) consider a somewhat more involved salvo attack on a uniform valued circular target. Specifically, they assume that each weapon has an error which is the sum of two independent components:

1. An offset aiming-point common to all n weapons, with a probability density function which is Gaussian, centered on the target, and with standard deviations $\sigma_x^2 = \sigma_y^2 = \sigma^2$.
2. An aiming-error independent from weapon to weapon, with a probability density function which is uniform inside a circle of radius D centered on the offset aiming-point.

They evaluate $E_n(f)$, the expected fraction of the target destroyed, by numerical methods. Jarnagin and DiDonato (1965) present their results in an extensive set of graphs.

2.2.3 Offense Can Place All Weapons Exactly

For the sake of completeness, the reader should be aware of a body of mathematical literature which was originally developed for purposes unrelated to weapons attacking undefended targets, but which may have a bearing upon the latter problem. Specifically, assume that the offense can place his n weapons on the uniform-valued circular target without any aiming errors whatsoever. How shall he place his weapons on the target in order to maximize the fraction of value destroyed? Assuming that $K > R$, two extreme cases can be distinguished:

1. if $n \leq n_0(K/R)$, the offense can place all his weapons completely within the target boundary without any weapon overlapping any other;
2. if $n \geq n_1(K/R)$, the offense can cover every point of the target with at least one weapon.

In the literature, these are referred to as packing and covering problems, respectively.

On pages 67 and 93-4 of his book, Fejes Tóth (1953) gives bounds for packing and covering rather general regions with identical circles. These may be specialized to yield bounds for n_0 and n_1:

$$(\pi/2\sqrt{3})\left((K/R) - 1\right)^2 < n_0(K/R) < (\pi/2\sqrt{3})(K/R)^2$$

$$(2\pi/3\sqrt{3})(K/R)^2 < n_1(K/R) \leq (2\pi/3\sqrt{3})(K/R)^2 + (4/\sqrt{3})(K/R) + 1 \; .$$

Unfortunately, it is not easy to determine the exact values of $n_0(K/R)$ and $n_1(K/R)$ unless K/R is quite small. Since n can only go up by integral jumps, it is more convenient to turn the problem around and specify the largest target circle which can be completely covered by n weapons, and the smallest target circle into which n weapons can be packed without overlap.

The largest target circle which can be completely covered by two weapons has a radius $K = R$; thus two weapons have no advantage over one in covering a circle. Three weapons can be placed at the midpoints of the sides of an equilateral triangle inscribed in a circle of radius $K = 2R/\sqrt{3}$, and four weapons at the midpoints of the sides of a square inscribed in a circle of radius $K = \sqrt{2}\,R$. These placements clearly cover their respective circles. The corresponding K are the largest for which complete coverage is possible; however the proof is less trivial than it might appear.

However, this sort of construction is no longer available for five weapons. In fact, the five-weapon coverage problem is not easy to solve. Intuitively, one might think that the best coverage is obtained by placing the five circles at the vertices of a pentagon of such a size that all five circles pass through the center of the pentagon; the radius of the circle covered by this arrangement is $K = 2R \cos 36^\circ = 1.6180R$. However, Neville (1915) proved that the radius of the largest target circle coverable by five weapons is $K = 1.6409R$; to find this constant, it was necessary for him to solve four nonlinear equations in four unknowns. (Specifically, each equation consists of the sums and differences of circular functions; the unknowns appear as linear functions of the arguments.) The five-circle covering problem was a popular diversion at English country fairs and shows of more than 50 years ago. For a nominal entrance fee, the player was invited to drop 5 circular discs on a larger circle so that it was completely covered; if he succeeded, he was awarded a much larger sum of money. Of course, the operator of the concession eliminated the "obvious" solution by setting the radius of the larger circle equal to (say) 1.63R.

The smallest target circle which can contain two non-overlapping weapons has a radius K = 2R. The corresponding radii of the smallest target circles containing three, four, five, six and seven weapons are $R(1+2/\sqrt{3})$, $R(1+\sqrt{2})$, $R(1+\sec 54°)$, 3R and 3R, respectively. For n greater than seven, the solution to the packing problem is not known.

2.3 EXPECTED FRACTIONAL DAMAGE OF A GAUSSIAN TARGET

In the preceding section the value of the target was considered to be spread uniformly throughout the interior of a circle of radius K. It is perhaps more realistic to consider targets which have greater value per unit area concentrated near the center and less near the edges. Such a target can be approximated by a Gaussian value density function:

$$T\left(x_t,y_t\right) = \left(1/2\pi\sigma_T^2\right) \exp\left(-\left(x_t^2 + y_t^2\right)/2\sigma_T^2\right) \ .$$

2.3.1 One Weapon Impact, Gaussian Aiming Error

The expected fraction of total value destroyed by a single weapon is equal to

$$E(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T\left(x_t,y_t\right) d\left(x-x_t,y-y_t\right) p(x,y)\, dx_t\, dy_t\, dx\, dy \ .$$

Assume the cookie-cutter damage function of previous sections, but a Gaussian probability density function of weapon impact-points. Then the above simplifies to

$$E(f) = \iint_{\left(x^2+y^2\right)^{1/2} \le R} p_1(x,y)\, dx\, dy \ ,$$

where

$$p_1(x,y) = \frac{1}{2\pi\left(\sigma_x^2 + \sigma_T^2\right)^{1/2}\left(\sigma_y^2 + \sigma_T^2\right)^{1/2}}$$

$$\cdot \exp\left(-\frac{\left(x-x_0\right)^2}{2\left(\sigma_x^2 + \sigma_T^2\right)} - \frac{\left(y-y_0\right)^2}{2\left(\sigma_y^2 + \sigma_T^2\right)}\right)$$

To see why this is so, remember that one can replace the Gaussian target with a point target having its location determined by a random variable with a Gaussian probability density function. Because the sum of two independent Gaussian random variables is again Gaussian, the position of the target relative to the impact-point of the weapon is distributed according to $p_1(x,y)$. In other words, all the derivations in Sections 2.1.1 - 2.1.4 can be immediately applied to the Gaussian target if the probability of target destruction is reinterpreted as the expected fraction of total value of the target destroyed, with σ_x^2 and σ_y^2 replaced by $\sigma_x^2 + \sigma_T^2$ and $\sigma_y^2 + \sigma_T^2$ respectively.

2.3.2 Multiple Weapon Impacts, Gaussian Aiming Error

However, this simplicity does not carry over to the problem of determining $E_n(f)$ when n weapons are directed at the target. To simplify the problem, assume that the weapon impact-points have a circular Gaussian probability density function centered on the target, and set $\sigma_x^2 = \sigma_y^2 = \sigma^2$ as before. From McNolty (1967), one obtains the expected fraction of total value of the target destroyed:

$$E_n(f) = \left(1/2\pi\sigma_T^2\right)\int_0^{2\pi}\int_0^{\infty} r\exp\left(-r^2/2\sigma_T^2\right)$$

$$\cdot\left(1 - \left(1 - P(R/\sigma, r/\sigma)\right)^n\right) dr\, d\theta$$

$$= 1 - \frac{1}{\sigma_T^2\sigma^{2n}}\int_0^{\infty} r\exp\left(-\frac{nr^2}{2\sigma_T^2} - \frac{r^2}{2\sigma^2}\right)$$

$$\cdot\left\{\int_R^{\infty} t\exp\left(\frac{-t^2}{2\sigma^2}\right) I_0\left(\frac{tr}{\sigma^2}\right) dt\right\}^n dr\ .$$

As before, $E_n(f) \le 1 - \left(1 - E_1(f)\right)^n$. Actually, McNolty gives $E_n(f)$ when there is also an offset aiming error; but this additional complication has been omitted. If one talks in terms of a point target having an unknown location given by a Gaussian probability density function, McNolty puts down for the record three other probability expressions:

a. The probability that exactly m weapons out of n will impact within distance R of the target.

b. The expected number of weapons required to obtain exactly one impact within a distance R of the target.

c. The probability that n or fewer weapons will be required to obtain exactly one impact within a distance R of the target.

The latter two quantities are useful only if the offense has a shoot-look-shoot capability; that is, if he can observe whether or not the target has been destroyed before committing another weapon. (It is hard to see how this would happen when the offense cannot even locate the target; but the target could perhaps be a hidden radio transmitter.) It is difficult to see why McNolty included the last-mentioned probability, for it is identical to $E_n(f)$: both are equal to 1 - Pr(first n weapons all miss the target). If the offset aiming error is zero the expected number of weapons required to destroy the point target is

$$E(n) = \frac{\sigma^2}{\sigma_T^2} \int_0^\infty r \exp\left(-\frac{r^2}{2\sigma_T^2} + \frac{r^2}{2\sigma^2}\right) \cdot \left\{\int_0^R t \exp\left(-\frac{t^2}{2\sigma^2}\right) I_0\left(\frac{tr}{\sigma^2}\right) dt\right\}^{-1} dr .$$

In general, this is larger than the inverse of the single-weapon probability of target destruction, $1 - \exp\left(-R^2/2\left(\sigma^2 + \sigma_T^2\right)\right)$.

Apparently, the quantity $E_n(f)$ has never been tabulated for targets having a Gaussian value density function. The following short table was obtained by simulation:

Salvo Size n = 2

R/σ_T	σ/σ_T = 0.25	0.5	0.75
0.6	.224	.235	.201
1.2	.574	.582	.563
1.8	.834	.832	.814
2.4	.949	.951	.942
3.0	.990	.988	.985

Salvo Size n = 5

R/σ_T	σ/σ_T = 0.5	0.75	1.0
0.4	.225	.214	.187
0.8	.521	.551	.516
1.2	.728	.773	.763
1.6	.865	.892	.893
2.0	.945	.956	.957

Salvo Size n = 20

R/σ_T	σ/σ_T = 0.25	0.5	0.75	1.0	1.5
0.2	.151	.209	.201	.176	.113
0.4	.280	.447	.531	.499	.386
0.6	.404	.595	.716	.740	.640
0.8	.528	.706	.817	.864	.828
1.0	.639	.792	.882	.924	.924
1.2	.735	.856	.926	.958	.970
1.4	.822	.900	.953	.975	.984

These tables were derived by the following Monte Carlo procedure. Obtain from a table of random Gaussian deviates (of mean zero and variance unity) $2n + 2$ independent values $X, Y, x_1, y_1, \ldots, x_n, y_n$. Calculate the quantity

$$\min_{1 \le i \le n} \left(\left(X + (\sigma/\sigma_T) x_i \right)^2 + \left(Y + (\sigma/\sigma_T) y_i \right)^2 \right)^{1/2} .$$

Repeat this procedure 5000 times and arrange these minima in order from smallest to largest. The table entry is the fraction of the minima that are less than R/σ_T; i.e., when n was 2 and σ/σ_T was 0.25, 1120 of the 5000 minima were less than 0.6. As is typical of a Monte Carlo approximation, the accuracy is low. The standard deviation of the estimate indicates that the above values are accurate to about two or three places.

Those interested in other approximations for $E_n(f)$ can use an approximate method developed by Groves and Smith (1957). These authors outline a crude numerical integration of $E_n(f)$. They approximate the Gaussian value density function with a set of ten rings each containing one-tenth of the target value. The ith ring has a radius, r_i, such that

$$\frac{i}{10} - \frac{1}{20} = \int_0^{r_i} \left(r/\sigma_T^2\right) \exp\left(-r^2/2\sigma_T^2\right) dr \ .$$

(This is quickly read off from the curve labeled $b/\sigma_A = 0$ in their Figure 1.) For each ring, the expected fraction of value destroyed, $E_n\left(f,r_i\right) = 1 - \left(1 - P\left(R/\sigma, r_i/\sigma\right)\right)^n$ is readily calculated with the aid of a table of $P(R/\sigma, r/\sigma)$ as defined in Section 2.1.2. Finally,

$$E_n(f) = \sum_{i=1}^{10} E_n\left(f,r_i\right)/10 \ .$$

Note that Groves and Smith actually supply graphs for a more general problem: an offset aiming point, b.

2.3.3 Non-Gaussian Aiming Error

Somewhat more complicated attacks on Gaussian-valued targets have been considered in the literature. McNolty (1962) considers an attack by one weapon which has an error which is the sum of two independent components:

1. An offset aiming-point with a gamma, beta, or Maxwell-Boltzmann probability density function.
2. An aiming-error with a probability density function which is circular Gaussian with mean at the offset aiming-point and variance $\sigma_x^2 = \sigma_y^2 = \sigma^2$.

Unfortunately, the three resulting expressions for E(f) are quite involved and laborious to compute. No tables or approximations are available. In a later reference, McNolty (1967) derives $E_n(f)$

for attacks by n weapons in which each weapon has the same offset error (1) drawn from the probability density function

$$g(r/\lambda,\alpha) = \left(2/\Gamma(\lambda)\right) \ (\lambda/\alpha)^{\lambda} \ r^{2\lambda-1} \ \exp\left(-\lambda/\alpha r^2\right) \ ,$$

but an independent circular Gaussian aiming error (2).

Holla (1970) derives complicated expressions for the probability of target kill under the assumption that the offset aiming-point is a random value drawn from the noncentral chi-square probability density function (the square root of this random variable is distributed according to the probability density function discussed in Section 2.1.2). Holla claims that for certain special cases the kill probability can be easily ascertained. It is unclear whether any of McNolty's or Holla's models of aiming-point error is likely to arise in a real-world bombing problem.

In an attempt to derive a more tractable expression for $E_n(f)$, Duncan (1964) somewhat modified the assumptions of the problem. He postulates a target having a circular Gaussian value density function with variance σ_T^2, but replaces the circular Gaussian probability density function of weapon impact-points with a uniform probability density function of impact-points inside a circle of radius $D > R$ centered on the target. Specifically, he ignores edge-effects by assuming that the entire lethal area of each weapon, πR^2, falls at random within the circle of radius D. The expected fraction of total value destroyed is given by

$$E_n(f) = \left(1 - \exp\left(-D^2/2\sigma_T^2\right)\right) \left(1 - \exp\left(-nR^2/D^2\right)\right) \ .$$

Duncan then determines that value of D which maximizes $E_n(f)$. Differentiate $E_n(f)$ with respect to D, set the result to zero, and solve for D. One obtains the optimum value of D:

$$\left(D_{opt}/\sigma_T\right)^2 = \sqrt{2n}\left(R/\sigma_T\right) \ .$$

The corresponding expectation is

$$E_n(f) = \left(1 - \exp\left(-\sqrt{n/2}\left(R/\sigma_T\right)\right)\right)^2 \ .$$

2.3.4 Offense Can Place All Weapons Exactly

Suppose that the offense can place his n weapons on the Gaussian-valued target without any aiming errors whatsoever. How should he place his weapons on the target in order to maximize the expected fraction of value destroyed? If he has only one weapon, the answer is obvious: place it at the center of the target. The corresponding $E_1(f)$ is

$$E_1(f) = 1 - \exp\left(-R^2/2\sigma^2\right) .$$

For two weapons, the problem of positioning has been solved by Marsaglia (1965). If the destruction radius of a weapon is equal to $R = R_0\sigma_T$, he shows that one should place the two weapons on opposite sides of the center of the target, each offset a distance $d = d_0\sigma_T$ from the center, where

$$\int_{-d_0/R_0}^{1} \left(1-y^2\right)^{-1/2} y\, e^{-d_0 R_0 y}\, dy = 0 .$$

He does not give the corresponding value of $E_2(f)$. For three or more weapons, the problem remains unsolved.

Gilliland (1966) has examined the easier problem of maximizing the expected fraction of value destroyed when the centers of the weapons must all be at least 2R apart (no overlap allowed). If $R/\sigma_T \leq 1$, two weapons should be placed on opposite sides of the origin, each offset a distance R from the origin: if $R/\sigma_T \leq 2/(1+\sqrt{3})$, three weapons should be placed at the vertices of an equilateral triangle with sides 2R centered on the origin; if $R/\sigma_T \leq 1/\sqrt{3}$, four weapons should be placed at the vertices of a rhombus with sides 2R and smaller diagonal 2R centered on the origin.

2.3.5 A Generalization of the Gaussian Target

McNolty (1968a, 1968b) introduces a family of targets having a value density with non-uniform phase; the Gaussian target (with uniform phase density) is a special case. Specifically, he assumes that the probability density function of target value with respect to the origin of coordinates has the following radial component:

$$g(r) = \left((\lambda r)^Q/\gamma^{Q-1}\right) \exp\left(-\left(\gamma^2/2\lambda\right) - (\lambda/2)r^2\right) I_{Q-1}(\gamma r) ,$$

where $I_n(r)$ is a modified Bessel function of the first kind of order n. Note that g(r) is a function of the three parameters $\lambda > 0$, $Q > 0$ and $\gamma \geq 0$. If one makes the additional assumption that the x-component and the y-component of the target value density are independent and have identical probability density functions, then one can derive the phase component of the probability density function of target value:

$$h(\theta) = \frac{1}{2}\left(|\sin 2\theta/2|^{Q-1}\right) \exp\left(-\gamma^2/2\lambda\right) \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} (\gamma/2)^{2i+2j}$$

$$\cdot\ \Gamma(Q+i+j)\sin^{2i}\theta\ \cos^{2j}\theta/\lambda^{i+j} i!j!\ \Gamma\left((Q/2)+i\right)\ \Gamma\left((Q/2)+j\right)\ .$$

Transforming from polar to rectangular coordinates, the x-component of the probability density function is

$$f(x) = 2^{Q/4-3/2}\left(|x|^{Q/2}/\gamma^{Q/2-1}\right)\lambda^{Q/2}$$

$$\cdot\ \exp\left(-\left(\gamma^2/4\lambda\right) - (\lambda/2)x^2\right) I_{Q/2-1}\left(\gamma(|x|/\sqrt{2})\right)\ .$$

If one assumes a cookie-cutter damage function and a circularly symmetric Gaussian probability density function of weapon impact-points centered at the origin of coordinates and with variance σ^2, then the expected fraction of total target value destroyed is

$$E(f) = \left(\lambda\sigma^2/\left(1+\lambda\sigma^2\right)\right)^Q \exp\left(-\gamma^2/2\lambda\right)\sum_{i=1}^{\infty}\gamma\left(i+1, R^2/2\sigma^2\right)$$

$$\cdot\ {}_1F_1\left(Q+i, Q, \gamma^2\sigma^2/2\left(1+\lambda\sigma^2\right)\right)/i\left(1+\lambda\sigma^2\right)^i\ i!B(i,Q)\ ,$$

where ${}_1F_1(x,y,z)$ is the confluent hypergeometric function, $\gamma(x,y)$ the incomplete gamma function, and $B(x,y)$ the Beta function $\Gamma(x)\Gamma(y)/\Gamma(x+y)$. The corresponding expected fraction destroyed by a salvo of n weapons, $E_n(f)$, is not known.

One can gain a little insight into McNolty's generalized target value densities by considering special cases. If one sets $Q = 1$, $\gamma = 0$ and $\lambda = 1/\sigma_T^2$, one has a Gaussian-valued target centered at the origin with variance σ_T^2, and the expected fraction of total target value destroyed reduces to the expression derived at the start of this section:

$$E(f) = 1 - \exp\left(-R^2/2\left(\sigma^2 + \sigma_T^2\right)\right)\ .$$

The parameter λ is a scale parameter, inversely proportional to the area covered by the target (more accurately, to that area with a value density greater than a certain minimum amount). The parameter γ is a non-centrality parameter, denoting the distance from the center of the target value density to the origin of coordinates. The parameter Q specifies the non-uniformity of the phase. For example, if $Q = 3/2$, the value density resembles a four-leaf clover (one petal in each quadrant); if $Q = 1/2$, the value density resembles the letter X (aligned with the two axes).

The weapons analyst who has a good knowledge of the value density of his target (and who plans to fire only one weapon at the target) may wish to approximate his target using one of McNolty's models. However, if his knowledge of the target is rather vague, he may be better advised to assume the target both uniform-valued circular and Gaussian, and determine how much difference exists between these two simple models.

2.4 THE DIFFUSED GAUSSIAN DAMAGE FUNCTION

By far the most frequently-used damage function has been the cookie-cutter: $d\left(x-x_t, y-y_t\right) = 1$ if $\left(\left(x-x_t\right)^2 + \left(y-y_t\right)^2\right)^{1/2} \leq R$ and $d\left(x-x_t, y-y_t\right) = 0$ otherwise. Its wide popularity seems to be due to conceptual simplicity; furthermore, Weidlinger (1962) has developed a physical argument involving cumulative damage from repeated impacts which appears to justify its use. However, the previous three sections show that a cookie-cutter damage function frequently leads to mathematical difficulties. Analytic expressions for probability of target destruction or expected target value destroyed usually cannot be obtained, and one must resort to tables or approximations. Is it possible that other damage functions might lead to easier mathematics?

2.4.1 Alternative Damage Functions

A wide variety of different damage functions have, in fact, been proposed. In general, a damage function is circularly symmetric (a function of $r = \left(\left(x-x_t\right)^2 + \left(y-y_t\right)^2\right)^{1/2}$ alone) and non-increasing from one to zero along any radius outward from the origin. (However, there have been exceptions to even these modest restrictions — for example, Bronowski and Neyman (1945) analyzed the coverage-properties of a rectangular cookie-cutter.) The following have been proposed by Guenther and Terragno (1964), Hunter (1967) and McNolty (1965):

$$d(r) = \exp\left(-r^2/2b^2\right)$$

$$d(r) = \exp(-br)\ , \qquad b > 0$$

$$d(r) = \left(\max\left(b^2-r^2, 0\right)\right)^{1/2}/b\ , \qquad b > 0$$

$$d(r) = \max(1-r/b,0)\,, \qquad b > 0$$

$$d(r) = \min\left(1,\ (r-R+1)^{-2}\right)$$

$$d(r) = \min\left(1,\ \exp\left(-\left(r^2-R^2\right)/2b^2\right)\right)$$

$$d(r) = \sum_{i=1}^{k} c_i \exp\left(-r^2/2b_i^2\right), \qquad c_i > 0,$$

$$d(r) = 1\,, \qquad 0 \le r \le R:\ d(r) = c\ \exp\left(-r^2/2b^2\right), \qquad c > 0, \qquad r > R$$

Of these alternative damage functions, the first is the most frequently encountered in the literature. For convenience, it is called the diffused Gaussian damage function. (Morgenthaler (1961) calls it the diffused exponential, but his terminology seems more appropriate for the second function.)

For the reader who desires a damage-function somewhere between the cookie-cutter and the diffused Gaussian, the Operations Evaluation Group (1959) and Galiano and Everett (1967) have proposed a family of damage functions expressible in terms of the Poisson distribution:

$$d_i(r) = \exp\left(-ir^2/2b^2\right) \sum_{j=0}^{i-1} \left(ir^2/2b^2\right)^j/j!$$

$$= 1 - P_{i-1}\left(ir^2/2b^2\right)\ .$$

Note that $d_1(r)$ is the diffused Gaussian damage function and $d_\infty(r)$ is the cookie-cutter. If one assumes a target with a circular Gaussian value distribution having a variance σ_T^2, and a circular Gaussian probability density function of weapon impact points centered on the target with variance σ^2, Galiano and Everett (1967) show that the expected fraction of target value destroyed by one weapon is

$$E_1(f) = 1 - \left(i\left(\sigma^2 + \sigma_T^2\right)\Big/\left(b^2 + i\left(\sigma^2 + \sigma_T^2\right)\right)\right)^i\ .$$

The Operations Evaluation Group (1959) provides a graph of $E_1(f)$ for i equal to two, generalizing it to an offset aim point.

The rest of this section (with two exceptions: Read (1971) and McNolty (1965) at the end of Section 2.4.2) is restricted to the diffused Gaussian damage function.

2.4.2 One Weapon Impact, Various Target Characteristics

Assume that the probability density function of weapon impact points is Gaussian centered on (x_0,y_0) and with variances σ_x^2 and σ_y^2; assume also that the target is centered on (0,0) and has a circular Gaussian value distribution with variance σ_T^2. The expected fraction of the target value destroyed by a single weapon, $E_1(f)$, is determined by a characteristic-function argument in McNolty (1967):

$$E_1(f) = \frac{b^2 \exp\left(-\frac{1}{2}\left(\frac{x_0^2}{b^2+\sigma_x^2+\sigma_T^2} + \frac{y_0^2}{b^2+\sigma_y^2+\sigma_T^2}\right)\right)}{\left(b^2+\sigma_x^2+\sigma_T^2\right)^{1/2}\left(b^2+\sigma_y^2+\sigma_T^2\right)^{1/2}} .$$

$E_1(f)$ also represents the probability of destruction of a point target located at (0,0) if one sets $\sigma_T^2 = 0$ in the above expression. Note the great analytic simplicity gained by changing from a cookie-cutter to a diffused Gaussian damage function; no longer does one have to deal with tabulated functions such as $P\left(R/\sigma, r_0/\sigma\right)$ and $P\left(R/\sigma_{max}, c\right)$.

If the target consists of two or more isolated points, how should a single weapon be aimed in order to maximize the probability of target destruction? Consider the simplest possible configuration: two point targets located at (-d,0) and (d,0). Assume that the probability density function of the impact-point of the weapon is circular Gaussian with $\sigma_x^2 = \sigma_y^2 = \sigma^2$. Generalizing an argument used in Operations Evaluation Group (1959), it is not difficult to show that the probability of destroying both targets with a single weapon aimed at (d-c,0), $0 \le c \le 2d$, is

$$P(\text{both}) = \frac{b^2}{b^2+2\sigma^2} \exp\left\{-\frac{d^2}{b^2} - \frac{(d-c)^2}{b^2+2\sigma^2}\right\} .$$

This is maximized by setting c = d; in other words, the optimum aim point is at the origin. However, it appears quite difficult to find the aim point which maximizes the corresponding probability of destroying at least one target.

When $\sigma_x^2 = \sigma_y^2 = \sigma^2$ and $x_0 = y_0 = 0$ (a circular Gaussian distribution of impact-points centered on the Gaussian-valued target), McNolty (1965) also derives $E_1(f)$ for the diffused exponential damage function $d(r) = \exp(-br)$:

$$E_1(f) = \sum_{i=0}^{\infty} (-1)^i b^i \left(2\left(\sigma_T^2 + \sigma^2\right)\right)^{i/2} \Gamma\left(1 + (i/2)\right)/i! \ .$$

When $\sigma_x^2 = \sigma_y^2 = \sigma^2$ and $x_0 \neq 0$, $y_0 \neq 0$ (a circular Gaussian distribution of impact-points offset from the Gaussian-valued target), Read (1971) derives $E_1(f)$ for the generalized cookie-cutter damage function $d(r) = 1$ $(0 \leq r \leq R)$, $d(r) = c \exp\left(-r^2/2b^2\right)$, $(r > R)$. The expression is a very complicated function of R, b, c, σ^2, σ_T^2, and $\left(x_0^2 + y_0^2\right)$, and the reader is referred to his paper for details.

2.4.3 Gaussian Target, More Than One Impact

What if more than one weapon is directed against the Gaussian target? Grubbs (1968) introduces the most general model; he assumes that each weapon impact-point is the sum of two independent random variables:

1. An offset aiming-point (common to all n weapons) with a Gaussian probability density function centered on the target, and with variables σ_{xa}^2 and σ_{ya}^2.
2. An aiming-error (independent from one weapon to the next) with a Gaussian probability density function centered at the offset aiming-point $\left(a_x, a_y\right)$ and with variances σ_x^2 and σ_y^2.

The expected fractional value destroyed of the Gaussian target is approximated by

$$E_n(f) \sim \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} b^{2i} \left(\left(b^2 + \sigma_x^2\right)\left(b^2 + \sigma_y^2\right)\right)^{-(i-1)/2}$$

$$\cdot \left\{\left(\left(b^2 + \sigma_x^2\right) + i\left(\sigma_T^2 + \sigma_{xa}^2\right)\right) \left(\left(b^2 + \sigma_y^2\right) + i\left(\sigma_T^2 + \sigma_{ya}^2\right)\right)\right\}^{-1/2}$$

$$\cdot \left(1 - \exp\left(-A_x^2/2\right)\right)^{1/2} \left(1 - \exp\left(-A_y^2/2\right)\right)^{1/2}$$

where

$$A_x^2 = k^2\left(b^2 + \sigma_x^2 + i\left(\sigma_T^2 + \sigma_{xa}^2\right)\right)\Big/\sigma_T^2\left(b^2 + \sigma_x^2 + i\sigma_{xa}^2\right)$$

and

$$A_y^2 = k^2\left(b^2 + \sigma_y^2 + i\left(\sigma_T^2 + \sigma_{ya}^2\right)\right)\Big/\sigma_T^2\left(b^2 + \sigma_y^2 + i\sigma_{ya}^2\right) .$$

This will be abbreviated

$$E_n(f) \sim \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} H_i .$$

The latter expression brings out the point that this approximation to $E_n(f)$ behaves like the binomial series, which is notoriously ill-suited to computation. The partial sums of this series may oscillate in sign, initially with increasing magnitude, until at some value of the summation index the magnitude begins to decrease. To get around these difficulties, Breaux and Mohler (1971) have developed three transformed series based on Jacobi polynomials that are well-adapted to computation and require perhaps fewer than one-half of the number of terms required by the binomial series. Unfortunately, the Jacobi polynomial coefficients are much more complicated than the binomial ones. Grubbs, Breaux and Coon (1971) suggest for the purposes of standardization that the simplest of the Jacobi polynomials be used:

$$E_n(f) \sim \sum_{j=1}^{n} a_{jn} \sum_{i=1}^{j} A_{ji} H_i$$

where

$$A_{ji} = 1 \qquad \text{for} \qquad i = 0 ,$$

$$A_{ji} = \sum_{k=1}^{i} (j+k+\alpha+\beta)(-j+k-1)/(\alpha+k)k \qquad \text{for} \qquad i = 1,2,\ldots,j ,$$

$$a_{jn} = \binom{n}{j} \frac{(2j+\alpha+\beta+1)\ \Gamma(j+\alpha+\beta+1)\ \Gamma(j+\alpha+1)\ \Gamma(n+\beta+1)}{\Gamma(1+\alpha)\ \Gamma(n+j+\alpha+\beta+2)\ \Gamma(j+\beta+1)} .$$

As a consequence of numerical experimentation, α was chosen to be .99 and β to be 0. For computational purposes, Grubbs, Breaux and

Coon (1971) use a slightly modified formula in which a_{jn} is expressed in terms of $a_{(j-1)n}$.

Guenther (1966) derives the same equation for $E_n(f)$ assuming that $\sigma_{xa} = \sigma_{ya}$ and $\sigma_x = \sigma_y$. If in addition there is no offset aiming point distribution $\left(\sigma_{xa} = \sigma_{ya} = 0\right)$, then the equation simplifies to

$$E_n(f) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \rho^i/(1+i\lambda) ,$$

where $\rho = b^2/\left(b^2+\sigma^2\right)$ and $\lambda = \sigma_T^2/\left(b^2+\sigma^2\right)$. Breaux (1968) shows how this can be reduced to

$$E_n(f) = 1 - B_\rho(1/\lambda, n+1)/\lambda\rho^{1/\lambda} ,$$

where $B_x(p,q)$ is the Incomplete Beta Function, which may be calculated from $I_x(p,q) = B_x(p,q)/B(p,q)$, tabulated by Pearson (1934). The above transformation also follows easily from formula 26.5.6 in Abramowitz and Stegun (1964).

2.4.4 Uniform Circular Target

Consider now the uniform-valued circular target of radius K centered at (0,0) instead of the Gaussian-valued target. Morgenthaler (1961) and Guenther and Terragno (1964) both derive the expected fraction of the target destroyed by a single weapon, assuming that the impact point has a Gaussian distribution centered on $\left(x_0, y_0\right)$ and with variances σ_x^2 and σ_y^2. The result is

$$E_1(f) = 2\left(b^2/K^2\right) P ,$$

where P is the corresponding probability of destruction of a point target by a single weapon with cookie-cutter damage function, as analyzed in Sections 2.1.1-2.1.4. However, one must replace σ_x^2, σ_y^2 or σ^2 with $\sigma_x^2 + b^2$, $\sigma_y^2 + b^2$ or $\sigma^2 + b^2$ and R^2 with K^2 in the earlier formulas. For example, if $\sigma_x^2 = \sigma_y^2 = \sigma^2$, then

$P = P\left(K\Big/\left(\sigma^2+b^2\right)^{1/2}, \left(x_0^2 + y_0^2\right)^{1/2}\Big/\left(\sigma^2+b^2\right)^{1/2}\right)$; if $x_0 = y_0 = 0$,

$P = P\left(K\Big/\left(\sigma_{max}^2 + b^2\right)^{1/2}, \left(\sigma_{min}^2 + b^2\right)^{1/2}\Big/\left(\sigma_{max}^2 + b^2\right)^{1/2}\right)$; if both conditions hold, $P = 1 - \exp\left(-K^2/2\left(\sigma^2+b^2\right)\right)$.

If more than one weapon is directed against the uniform-valued target, Grubbs (1968) assumes the same model of weapon impact-points as before. Unfortunately, the resultant expected damage function cannot be integrated. In order to obtain a simple expression, Grubbs resorts to two approximations:

1. Instead of integrating over an elliptical region, he integrates over a rectangular region of equivalent area, and
2. He replaces the integral (from -x to x) of the Gaussian probability density function with the expression $\left(1 - \exp\left(-2x^2/\pi\right)\right)^{1/2}$.

The expected fraction of the target destroyed is, approximately,

$$E_n(f) \sim \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \left(2/iK^2\right) b^{2i} \left(\left(b^2+\sigma_x^2\right)\left(b^2+\sigma_y^2\right)\right)^{-(i-1)/2}$$

$$\cdot \left\{1 - \exp\left(-iK^2/2\left(b^2 + \sigma_x^2 + i\sigma_{xa}^2\right)\right)\right\}^{1/2}$$

$$\cdot \left\{1 - \exp\left(-iK^2/2\left(b^2 + \sigma_y^2 + i\sigma_{ya}^2\right)\right)\right\}^{1/2} .$$

This will be abbreviated

$$E_n(f) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} G_i .$$

As mentioned in Section 2.4.3, the latter expression behaves like the binomial series and is ill-suited to computation. Breaux and Mohler (1971) have developed three transformed series based on Jacobi polynomials; the simplest one (suggested by Grubbs, Breaux and Coon (1971) in the interest of standardization) is given below:

$$E_n(f) \sim \sum_{j=1}^{n} a_{jn} \sum_{i=1}^{j} A_{ji} G_i ,$$

where A_{ji} and a_{jn} have already been defined in Section 2.4.3. For computational purposes, Grubbs, Breaux and Coon use a slightly modified formula in which a_{jn} is expressed in terms of $a_{(j-1)n}$. The reader is referred to their article for comments on the comparative accuracy of the binomial and Jacobi approximations.

Guenther (1966) gives the integral for $E_n(f)$ assuming that $\sigma_{xa} = \sigma_{ya}$ and $\sigma_x = \sigma_y$ but cannot integrate it. If in addition there is no offset aiming-point distribution $\left(\sigma_{xa} = \sigma_{ya} = 0\right)$, then the integral can be easily evaluated:

$$E_n(f) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \rho^i \left(1 - \exp(-i\tau)\right)/i\tau \ ,$$

where $\rho = b^2/\left(b^2+\sigma^2\right)$ and $\tau = K^2/2\left(b^2+\sigma^2\right)$. Notice that Grubbs' binomial series approximation for $E_n(f)$ reduces to this exact expression when $\sigma_{xa} = \sigma_{ya} = 0$. In addition note the (not surprising) similarity between the above two formulas to those of the previous section.

When n is moderately large size (say 20 to 50, depending on the word length of the computer) this formula becomes subject to serious round-off problems. To meet this difficulty Breaux (1968) transforms the above into a numerically stable form. The resulting expression has the added advantage that $E_1(f), E_2(f), \ldots, E_n(f)$ are all available as partial sums of the same series:

$$E_n(f) = \sum_{i=1}^{n} \left(\left(1 - \rho \exp(-\tau)\right)^i - (1-\rho)^i\right)/i\tau \ .$$

Morgenthaler (1961) considers a more general problem in which each weapon has an individual aiming-point; the resulting expressions are so cumbersome that he must resort to simulation. He also replaces a salvo of n weapons with an annulus of appropriate area having its center distributed according to a circular Gaussian distribution centered on the target. Unless the salvo can be arranged so that none of the weapons "overlap" (i.e., unless the impact-points are at least 3b or 4b apart), this approximation is likely to be rather poor.

2.5 MATCHING THE ATTACK DISPERSION TO AN AREA TARGET

In Sections 2.2.1-2.4.4, it has been assumed that the attacker aims all of his weapons at the center of the area target. If the radius of the target is large with respect to the standard deviation of the impact-point probability density function, and it is also large with respect to the radius of the damage function, then the center of the target will be overkilled by the salvo and the edge of the target left untouched. This problem can be overcome by "spreading out" the salvo in some sense — either by scattering the aiming-points throughout the target area, or by retaining the single aiming-point and increasing the standard deviation of the impact-point probability density function. (The latter technique is seen in the design of a shotgun choke.) A somewhat less practical solution to the problem is the precise tailoring of an impact point probability density function to maximize the expected fraction (or expected value) of target destroyed. However, the latter problem is worth investigating in its own right because it provides an upper bound to the damage achieved by optimizing either the placement of aiming-points or the standard deviation of the impact-point distribution. These three approaches to the problem of matching attack dispersion to target size, then, are the subject of the next sections.

2.5.1 Multiple Aiming-Points

It is not hard to see that the problem of selecting the optimum set of aiming-points against an area target is an exceedingly difficult one. The work of Fejes Tóth (1953), Neville (1915), Marsaglia (1965) and Gilliland (1966) well illustrates the difficulties when there is no error in the placement of the weapons on an area target. To combine this work with that of (say) Grubbs, Breaux and Coon (1971) to obtain a reasonably realistic multiple aiming-point model appears to be impossible at the present time.

What, in fact, has been accomplished? Bressel (1971) has developed perhaps the most comprehensive model. Specifically, he considers an attack of n weapons against a rectangular (instead of circular) uniform-valued target. Each weapon has its own aiming-point specified in a rectangular grid. Each weapon has an error which is the sum of two independent components:

1. an aiming-point error (common to all n weapons), with a Gaussian probability density function centered on its individual aiming-point, and with variances σ_{xa}^2 and σ_{ya}^2,
2. an individual dispersion (independent from weapon to weapon) with a Gaussian probability density function with variances σ_x^2 and σ_y^2.

He assumes a diffused Gaussian damage function. The resultant equation giving the expected function of target damage is far too cumbersome to optimize with respect to the two components of aiming-point spacing. In order to illustrate the optimization procedure, he considers a special one-dimensional case: the determination of the correct spacing between the aiming-points of two weapons dropped by an airplane perpendicularly crossing a railroad track. Setting $\sigma_{xa} = \sigma_{ya} = 100$ feet, $\sigma_x = \sigma_y = 25$ feet, a target of 2 feet by 200 feet (effectively infinite), and the damage function parameter b = 25 feet, Bressel determines that the optimum weapon spacing is about 74 feet (each weapon should be aimed 37 feet from the target center). He plots the expected damage as a function of weapon spacing in order to demonstrate that a true maximum is achieved; for sufficiently small values of $\sigma_{xa} = \sigma_{ya}$ (20 feet or less in this example), the optimum spacing is actually zero (both weapons should be aimed at the center of the target).

Using the same model as Bressel, Sangal (1969) and Jaiswal and Sangal (1972) have derived expressions for the expected fraction destroyed of a rectangular target attacked by (a) three weapons in a string, compared with an equilateral triangle aiming-point pattern, (2) four weapons in a string, compared with a square aiming-point pattern, and (3) five weapons in a string, compared with a quincunx aiming-point pattern (i.e., the arrangement of the pips on a die). Unfortunately, the expressions are so lengthy that it is tedious to determine either the optimum spacing of a given pattern (a function of target dimensions, aiming-point and dispersion variances, and damage function spread), or the conditions under which one or another of the optimized aiming-point patterns should be used.

The Bessel model can be specialized by retaining the diffused Gaussian damage function but simplifying the weapon aiming-errors, i.e., setting $\sigma^2_{xa} = \sigma^2_{ya} = 0$ and $\sigma^2_x = \sigma^2_y = \sigma^2$. Hunter (1967) uses this model to calculate $E_n(f)$ for a Gaussian target, and Read (1971) uses the same model to calculate $E_n(f)$ for a uniform-valued circular target. Unfortunately, the resultant expressions are so complicated that it is impossible to find the optimum set of aiming-points.

2.5.2 The Maximum Expected Damage if the Attacker Selects the Variance of the Weapon Impact-Points

In this section, a cookie-cutter (i.e., all-or-nothing) damage function with radius R is assumed. Two types of area targets will be considered, a uniform-valued circular target and a Gaussian-valued one. As will be seen, no analytic procedures are known, and it is necessary to treat the problem empirically. For simplicity, a circular Gaussian distribution of weapon impact points will be considered. The attacker's problem is to determine the $\sigma_{opt} = \sigma_x = \sigma_y$ which maximizes $E_n(f)$.

Consider a uniform-valued circular target with radius K. The evaluation of $E_n(f)$ for a given value of σ has been discussed in Section 2.2.2; σ_{opt} may be found empirically from tables of $E_n(f)$. The process is simplified by the fact that for fixed n and for h any constant, $h\sigma_{opt} = f(hK, hR)$, so that σ_{opt}/K is a function of R/K and n alone. One can use the extensive tables of Jarnagin (1965) to determine the optimum standard deviation. This brief table has been compiled from that source:

	n = 5		n = 10		n = 20	
R/K	σ_{opt}/K	$E_n(f)$	σ_{opt}/K	$E_n(f)$	σ_{opt}/K	$E_n(f)$
0.05	–	–	0.29	0.024	0.33	0.047
0.1	0.31	0.047	0.36	0.090	0.39	0.165
0.2	0.38	0.168	0.42	0.293	0.45	0.471
0.3	0.40	0.327	0.45	0.514	0.50	0.728
0.4	0.42	0.492	0.48	0.702	0.53	0.882
0.5	0.43	0.642	0.50	0.835	0.55	0.958
0.6	0.42	0.766	0.50	0.919	0.56	0.987
0.7	0.39	0.861	0.48	0.965	–	–
0.8	0.35	0.928	–	–	–	–

It would be of some interest to determine the asymptotic behavior of σ_{opt} and its associated $E_n(f)$ as n approaches infinity and R approaches zero such that nR^2 is a constant.

Consider next a Gaussian-valued target with variance σ_T^2. The calculation of $E_n(f)$, given σ, was discussed in Section 2.3.2. Analogously to the uniform-valued case, σ_{opt}/σ_T is a function of R/σ_T and n alone.

Using the table in Section 2.3.2 for a salvo size of 20, one can conjecture that as n approaches infinity and R approaches zero so that $nR^2 = c$, the σ_{opt} is given by $\left(\sigma_{opt}/\sigma_T\right)^2 \sim 1.5\left(R/\sigma_T\right)$. It seems plausible to assume that σ^2_{opt} is roughly proportional to $\sqrt{n}$; therefore, one obtains

$$\left(\sigma_{opt}/\sigma_T\right)^2 \sim 0.34\,\sqrt{n}\,\left(R/\sigma_T\right) .$$

Note that this is of the same form as D_{opt} in Duncan (1964), considered in Section 2.3.3. What is the corresponding value of $E_n(f)$? Normalizing to $\sigma_T = 1$, one can show that the limiting maximum $E_n(f)$ is

$$\lim_{\substack{n\to\infty \\ R^2=c/n}} E_n(f) = 1 - \int_0^\infty r\exp\left(-r^2/2\right)\left(1 - P(R/\sigma, r/\sigma)\right)^n dr$$

$$= 1 - \sigma^2\left(2\sigma^2/c\right)\int_0^{c/2\sigma^2} r^{\sigma^2-1}\exp(-r)\,dr$$

$$= 1 - \left(2\sigma^4/c\right)\gamma\left(\sigma^2, c/2\sigma^2\right) ,$$

where γ is the incomplete gamma function. If one substitutes in $\sigma^2_{opt} = 0.34\sqrt{c}\,\sigma_T$ for σ^2 and uses Pearson (1951) to evaluate the integral, one obtains the approximate table:

c/σ^2_T	σ_{opt}/σ_T	$\lim E_n(f)$
2.5	0.733	0.455
5.0	0.872	0.635
7.5	0.965	0.738
10.0	1.037	0.804
12.5	1.096	0.849
15.0	1.148	0.882
17.5	1.193	0.906
20.0	1.233	0.924
22.5	1.270	0.937

2.5.3 An Upper Bound to the Expected Damage if the Attacker Selects Any Probability Density Function of Weapon Impact-Points

It would be quite interesting to find a method of determining the optimum p(x,y) among all possible impact-point probability density functions, instead of restricting oneself to Gaussian ones. This appears to be an intractable problem; however, Walsh (1956) treats a related problem that could often lead to satisfactory approximations. His method is applicable to targets of arbitrary value-structure, not only Gaussian or uniform ones. Furthermore, Walsh solves this problem for many dimensions; this section restricts Walsh's argument to the two-dimensional case. It is convenient to formulate Walsh's problem in terms of the original problem.

Letting, as usual, $d\left(x-x_t, y-y_t\right)$ be the probability that a point $\left(x_t, y_t\right)$ of the target is destroyed by a weapon impact at (x,y), define

$$P\left(x_t, y_t\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\left(x-x_t, y-y_t\right) p(x,y)\, dx\, dy \quad .$$

Thus, $P\left(x_t, y_t\right)$ is the probability that the point $\left(x_t, y_t\right)$ is destroyed by a weapon launched at the target. Let $T\left(x_t, y_t\right)$ be the density function of the fractional value of the target; that is, $T\left(x_t, y_t\right)$ integrated over a region gives the fraction of the total value contained in that region. It is convenient, although not necessary, to insist that T be a bounded function, so that the integral of T is continuous. Then the expected fraction of the target destroyed by n independent weapons is

$$E_n(f) = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(x_t, y_t) \{1 - P(x_t, y_t)\}^n \, dx_t \, dy_t \; .$$

The first problem mentioned in this section can now be precisely formulated: maximize $E_n(f)$ over possible densities $p(x,y)$, given the above expression for $P(x_t, y_t)$. Unfortunately, there seems to be no way to solve this highly nonlinear variational problem.

Walsh, however, is able to find, for a class of possible $P(x_t, y_t)$, that $P(x_t, y_t)$ which maximizes $E_n(f)$. It is easy to derive a constraint for $P(x_t, y_t)$ by integrating the formula defining $P(x_t, y_t)$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x_t, y_t) \, dx_t \, dy_t \\
&\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(x - x_t, y - y_t) \, p(x,y) \, dx \, dy \, dx_t \, dy_t \\
&\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(x - x_t, y - y_t) \, dx_t \, dy_t \right) dx \, dy \\
&\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,y) \cdot D \, dx \, dy \\
&\qquad = D \; ,
\end{aligned}$$

where D is the total lethality of a weapon:

$$D = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(x,y) \, dx \, dy \; .$$

One assumes that $D > 0$, since otherwise the attacker can accomplish nothing. For brevity, the subscripts t are henceforth dropped in the expression $P(x_t, y_t)$.

Walsh finds that $P(x,y)$ which maximizes $E_n(f)$, subject to the above constraint on $P(x,y)$. If a $p(x,y)$ exists which generates the optimal $P(x,y)$ found by Walsh, then clearly $p(x,y)$ is optimal in the sense of maximizing $E_n(f)$. However, it is possible (in fact, likely) that in a specific problem no such $p(x,y)$ exists. In such a situation, one must settle for a $p(x,y)$ which leads to an approximation of the optimal $P(x,y)$.

The problem to be solved can be restated slightly as follows: Find that $P(x,y)$ which minimizes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(x,y) \left\{1 - P(x,y)\right\}^n dx\, dy \ ,$$

subject to the constraints

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,y)\, dx\, dy = D \ , \qquad 0 \le P(x,y) \le 1 \ .$$

Because Walsh gives an incomplete derivation of the solution (citing Svesnikov (1948), a source not readily available), it is worthwhile giving a full derivation, using a simple Lagrange multiplier argument.

First, consider the following problem: Let $\lambda > 0$ and minimize

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{T(x,y) \left(1 - P(x,y)\right)^n + n\lambda P(x,y)\right\} dx\, dy \ ,$$

subject only to $0 \le P(x,y) \le 1$. Suppose that for some λ the $P_\lambda(x,y)$ solving this problem satisfies the additional constraint. Then clearly this $P_\lambda(x,y)$ solves the original problem, since the term $n\lambda P(x,y)$ has no effect on the minimization. This is the Lagrange multiplier principle.

To minimize the above integral, observe that there are no constraints connecting values of $P(x,y)$ at different points. Therefore, it is sufficient to minimize the integrand independently at each point. Thus one is led to the problem of minimizing

$$T(1-P)^n + n\lambda P \ ,$$

where T is a constant and $0 \le P \le 1$. Elementary calculus yields the solution $P = \max\left\{0,\ 1 - (\lambda/T)^{1/(n-1)}\right\}$. Hence, the solution is given by $P_\lambda(x,y) = \max\left\{0,\ 1 - \left(\lambda/T(x,y)\right)^{1/(n-1)}\right\}$. Observe that for each (x,y), $P_\lambda(x,y)$ is a continuous monotone nonincreasing function of λ.

To complete the Lagrange multiplier solution, it is necessary to determine λ so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_\lambda(x,y)\,dx\,dy$$

$$= \iint_{T(x,y)>\lambda} \left\{1 - \left(\lambda/T(x,y)\right)^{1/(n-1)}\right\} dx\,dy = D\ .$$

This integral is a continuous monotone nonincreasing function of λ. Its value is zero at $\sup T(x,y)$. As λ approaches zero the value of the integral approaches A, the total extent of the target. That is, A is the area (possibly infinite) of the region for which $T(x,y) > 0$. Thus the above equation has a unique solution in λ provided that $D < A$. But if $D \geq A$ the original problem is trivial, since a $P(x,y)$ can be chosen satisfying the constraints and for which $P(x,y) = 1$ whenever $T(x,y) > 0$; thus the entire target can be destroyed by a single weapon. (This fact is additional evidence for the lack of realism of the model used.) In the case of interest, therefore, the equation has a solution. In practice, however, solving for λ is not too simple. In general, analytic solutions will not exist, so that it will usually be necessary to employ a search procedure. However, this may not be too difficult if a computer is available, since it should not be necessary to determine λ very accurately, and since the continuity and monotonicity of the integral would simplify the search.

In any case, the solution may be summarized as follows. First, determine λ from the equation

$$\iint_{T(x,y)>\lambda} \left\{1 - \left(\lambda/T(x,y)\right)^{1/(n-1)}\right\} dx\,dy = D\ .$$

Then the optimum $P(x,y)$ is given by

$$P(x,y) = \begin{cases} 1 - \left(\lambda/T(x,y)\right)^{1/(n-1)} & \text{if} \quad T(x,y) \geq \lambda\ , \\ 0 & \text{if} \quad T(x,y) < \lambda\ . \end{cases}$$

The corresponding $E_n(f)$ is given by

$$E_n(f) = 1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(x,y) \left(1 - P(x,y)\right)^n dx\, dy$$

$$= 1 - \iint_{T(x,y) < \lambda} T(x,y)\, dx\, dy - \iint_{T(x,y) \geq \lambda} T(x,y)$$

$$\cdot \left(\lambda / T(x,y)\right)^{n/(n-1)} dx\, dy$$

$$= 1 - \iint_{T(x,y) < \lambda} T(x,y)\, dx\, dy - \iint_{T(x,y) \geq \lambda} \left(\lambda^n / T(x,y)\right)^{1/(n-1)} dx\, dy$$

$$= \iint_{T(x,y) \geq \lambda} \left\{T(x,y) - \left(\lambda^n / T(x,y)\right)^{1/(n-1)}\right\} dx\, dy \ .$$

It is instructive to carry out these calculations in the simple case of a uniform-valued region $\mathcal{N}$. Let the area of $\mathcal{N}$ be A; then

$$T(x,y) = \frac{1}{A} \ , \qquad (x,y) \in \mathcal{N} \ ,$$

$$T(x,y) = 0 \ , \qquad (x,y) \notin \mathcal{N} \ .$$

Thus the equation for λ becomes (if $0 < \lambda < 1/A$):

$$A \left\{1 - (\lambda A)^{1/(n-1)}\right\} = D \ ,$$

so

$$\lambda = \frac{(1 - D/A)^{n-1}}{A} \ .$$

Hence

$$P(x,y) = \begin{cases} 1 - \left(\dfrac{(1-D/A)^{n-1}}{A} \cdot A\right)^{1/(n-1)} = D/A & \text{if} \quad (x,y) \in \mathcal{N} \ , \\ 0 & \text{if} \quad (x,y) \notin \mathcal{N} \ . \end{cases}$$

The resulting $E_n(f)$ is given by

$$E_n(f) = A \left\{ \frac{1}{A} - \left(\frac{(1-D/A)^{n(n-1)}}{A^n} \cdot \frac{1}{A} \right)^{1/(n-1)} \right\}$$

$$= 1 - (1-D/A)^n \ .$$

Therefore, the optimum $P(x,y)$ is a uniform density over the whole target. However, it is not possible in general to find any $p(x,y)$ which leads to this distribution. To be specific, assume that N is a circular region and that the damage function is a circular cookie-cutter. Then it is easy to see intuitively that for any $p(x,y)$, either the resulting damage function overlaps the boundary or it approaches zero near the boundary. More formally, one sees from the definition of $P(x,y)$ at the beginning of this section that $P(x,y)$ is a continuous function, provided $p(x,y)$ is a bounded function. Even if $p(x,y)$ is not bounded, a solution is impossible. In fact, it can be seen that in general, whatever the nature of the target value and damage functions, a solution is unlikely.

Nevertheless, the given $E_n(f)$ provides an upper bound. Thus if one can find a $p(x,y)$ which gives an $E_n(f)$ close to that bound, one knows that $p(x,y)$ is a good approximation to the optimum result.

2.5.4 An Asymptotic Probability Density Function of Weapon Impact-Points for a Gaussian-Valued Target

When one attempts to apply Walsh's method to a Gaussian-valued $T\left(x_t,y_t\right)$ instead of a uniform one, one runs into mathematical difficulties immediately. To find λ, one must solve the equation

$$2\pi \int_a^b r \left(1 - \lambda^{1/(n-1)} \left(\left(1/2\pi\sigma_T^2\right) \exp\left(-r^2/2\sigma_T^2\right)\right)^{-1/(n-1)}\right) dr = \pi R^2 \ ,$$

where a and b are the values of r which satisfy $\lambda = \left(r/\sigma_T^2\right)\exp\left(-r^2/2\sigma_T^2\right)$. Thus any solution of the problem would certainly make extensive use of a computer.

However, if one allows n to approach infinity and R to approach zero in such a way that nR^2 remains equal to a constant c, then it is possible to determine even the impact-point probability density function which maximizes $E_n(f)$ in the limit. One derivation can be found in the second half of a paper by Galiano and Everett (1967); a different one is given in Duncan (1964). In the limit, the expected fraction of total value destroyed is

$$\lim_{\substack{n\to\infty \\ R^2=c/n}} E_n(f) = 1 - \left(1 + \left(c/\sigma_T^2\right)^{1/2}\right) \exp\left(-\left(c/\sigma_T^2\right)^{1/2}\right).$$

This is often referred to in the literature as the square-root law of target damage. The impact-point probability density function is circularly symmetric about the center of the target. At a distance r from the target center, $0 \le r \le \left(2\sigma_T\right)^{1/2} c^{1/4}$, the density is proportional to the quantity $\left(c/\sigma_T^2\right)^{1/2} - \left(r^2/2\sigma_T^2\right)$; for all $r > \left(2\sigma_T\right)^{1/2} c^{1/4}$, the density is zero. Obviously, this truncated probability density function is not Gaussian; the table below shows how much the offense gains (in the limit as n approaches infinity) by switching from the optimum Gaussian probability density to the unconstrained optimum probability density.

EXPECTED FRACTION OF TARGET DESTROYED

	$\lim E_n(f)$	
c/σ_T^2	Gaussian	Optimum
2.5	0.455	0.469
5.0	0.635	0.652
7.5	0.738	0.758
10.0	0.804	0.824
12.5	0.849	0.868
15.0	0.882	0.899
17.5	0.906	0.921
20.0	0.924	0.938
22.5	0.937	0.950

2.6 ESTIMATING THE PROBABILITY OF TARGET SURVIVAL/DESTRUCTION FROM IMPACT-POINT DATA

If the probability density function of weapon impact points, p(x,y), is completely specified, one may apply directly the probability calculations outlined in earlier sections. However, a more realistic situation is that the form of the probability density function is known but certain parameters are unknown. For instance, one may know that p(x,y) is a Gaussian distribution function but may know neither the mean $\mu = \left(\mu_x, \mu_y\right)$ nor the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}.$$

However, one is assumed to have a set of observations of actual weapon impacts $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$. How can one use this data to estimate the probability that a weapon will impact within a circle of radius R, or (conversely) estimate the radius of a circle within which a weapon will impact with probability P?

The reader is warned that the following sections draw rather extensively on material from mathematical statistics; however, he should be able to understand the results in this section without a detailed knowledge of the field.

2.6.1 Estimation of the Probability of Impact Within a Circle

Consider the problem of estimating the probability P that a single weapon will impact in a circle of radius R centered on a point target, given a series of observations of actual weapon impacts $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$.

It is clear that a relatively straightforward estimate of this probability is immediately available. If m out of the n quantities $\left(x_i^2 + y_i^2\right)^{1/2}$ are less than or equal to R, then one can estimate the probability P by

$$\hat{P} = m/n \quad .$$

The variance of this estimate is estimated using the binomial probability distribution:

$$\mathrm{var}(\hat{P}) = (m/n)\left(1 - (m/n)\right)(1/n) \quad .$$

However, it should be possible to make more complete use of the available information and obtain an estimate with as small a variance as possible. To accomplish this, one can apply a well-known theorem from statistics:

> Assume that X is a random variable with a probability density function $f(x,\theta)$, and assume that one wishes to estimate a function $g(\theta)$. If a statistic t for θ which is complete and sufficient exists, and if an unbiased estimate W of $g(\theta)$ is known, then the minimum-variance unbiased estimate for $g(\theta)$ is given by the expectation of W given t, $E(W|t)$.

Laurent (1957, 1962) has determined the minimum-variance unbiased estimate $\hat{P}$ under the assumption that neither μ nor Σ is known. The estimate given below is a somewhat simplified expression due to Kabe (1965). Let λ_1 and λ_2 be the eigenvalues of the sample covariance matrix S; that is, λ_1 and λ_2 are the solutions to the equation

$$\left(S_x^2 - \lambda\right)\left(S_y^2 - \lambda\right) - S_{xy}^2 = 0 \quad ,$$

where

$$S_x^2 = \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2/n \quad ,$$

$$S_y^2 = \sum_{i=1}^{n} \left(y_i - \bar{y}\right)^2/n \quad ,$$

$$S_{xy} = \sum_{i=1}^{n} \left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)/n \quad ,$$

$$\bar{x} = \sum_{i=1}^{n} x_i/n \quad \text{and} \quad \bar{y} = \sum_{i=1}^{n} y_i/n \quad .$$

Assume that $\lambda_1 > \lambda_2$. Then, assuming n is even,

$$\hat{P} = (2\pi)^{-1}\,(n-2)\left(\lambda_2/\lambda_1\right)^{1/2} \sum_{i=0}^{(n-4)/2} \binom{(n-4)/2}{i} B\left(i+1/2, 1/2\right)$$

$$\cdot \left(1 - \left(\lambda_2/\lambda_1\right)\right)^i B\left(i+1, (n/2) - i - 1\right) I_{\left(R^2/n\lambda_2\right)}\left(i+1, (n/2) - i - 1\right) \quad ,$$

where $B(p,q)$ is the Beta Function $\Gamma(p)\,\Gamma(q)/\Gamma(p+q)$, and where $I_x(p,q)$ is the Incomplete Beta Function tabulated in Pearson (1934):

$$B_x(p,q) = \left\{\int_0^x y^{p-1}\,(1-y)^{q-1}\,dy\right\}/B(p,q) \quad .$$

This function is also described on page 263 of Abramowitz and Stegun (1964). The variance of this estimate is unknown;

it is impossible to say how much gain has been achieved over the simple estimate $\hat{P} = m/n$.

If one is willing to assume that $\sigma_{xy} = 0$, Kabe (1968) derives minimum variance estimates for $\hat{P}$ under three conditions: Σ unknown; μ known; Σ known, μ unknown; both Σ and μ unknown. Unfortunately, these are given as bivariate integrals which are not easy to evaluate. For example, if Σ is unknown and μ is known; then the minimum-variance estimate of P is

$$\hat{P} = \left(B\left(1/2,(n-1)/2\right)\right)^{-2} \iint \left(\left(S_x-u^2\right)\left(S_y-v^2\right)\right)^{(n-3)/2} \cdot \left(S_x S_y\right)^{-(n-2)/2} \, du\, dv \ ,$$

where the integration is taken over the region $u^2 + v^2 \leq R^2$, and

$$S_x = \sum_{i=1}^{n} x_i \ , \qquad S_y = \sum_{i=1}^{n} y_i \ .$$

Kabe outlines how to evaluate this integral by means of a triple summation.

If one is willing to assume that p(x,y) is a circular Gaussian distribution with mean centered on the target, then the only unknown parameter is $\sigma_x^2 = \sigma_y^2 = \sigma^2$. The problem of determining the minimum-variance unbiased estimate of P, the probability that a weapon will land within a distance R of the point target, is considerably easier. Inselmann and Granville (1967) have carried out this computation using the same theorem as Laurent. They find that

$$\hat{P} = \frac{\left\{\left(2nS^2\right)^{2n-3} - \left(2nS^2-R^2\right)^{2n-3}\right\}(n-1)}{(4n-6)n^{n-1}\, 2n^{-2}\, S^{2n-2}}$$

where

$$S^2 = \sum_{i=1}^{n} \left(x_i^2 + y_i^2\right)/2n \ .$$

Again, it would be desirable to know the variance of $\hat{P}$.

2.6.2 Estimation of the Radius of a Circle Corresponding to a Given Impact Probability

This section considers the converse of the previous problem. As before, assume that p(x,y), the probability density function of weapon impact-points, is Gaussian with arbitrary mean μ and covariance matrix Σ. The problem is now to estimate the radius R of a circle centered on a point target at the origin of coordinates, given that the probability of target destruction is P, and given a series of weapon impacts $\left(x_1,y_1\right),\left(x_2,y_2\right),\ldots,\left(x_n,y_n\right)$. This can be readily done by first estimating μ and Σ by means of the unbiased sample moments

$$T_x^2 = \sum_{i=1}^{n} \left(x_i-\bar{x}\right)^2/(n-1)$$

$$T_y^2 = \sum_{i=1}^{n} \left(y_i-\bar{y}\right)^2/(n-1)$$

$$T_{xy} = \sum_{i=1}^{n} \left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)/(n-1)$$

$$\bar{x} = \sum_{i=1}^{n} x_i/n \quad \text{and} \quad \bar{y} = \sum_{i=1}^{n} y_i/n \ ,$$

next rotating the coordinate axes through an angle ϕ to get rid of T_{xy} $\left(\phi \text{ is the solution to the equation } \tan 2\phi = 2T_{xy}/\left(T_x^2 - T_y^2\right)\right)$, and finally identifying the remaining four transformed estimates with the (unknown) mean and variance of the probability density function of weapon impact-points. One can then calculate R implicitly using the methods of Section 2.1.

Blischke and Halpin (1966) derive the variance of this estimate of R:

$$\begin{aligned}\operatorname{var}(\hat{R}) = {} & \left(1/nC_0^2\right)\left(\sigma_x^2C_1^2 + 2\sigma_{xy}C_1S_1 + \sigma_y^2S_1^2\right) + \left(1/4(n-1)C_0^2\right) \\ & \cdot \left\{\left(RC_0 - \mu_xC_1 - \mu_yS_1\right)^2 + \left(RS_2 - \mu_xS_1 - \mu_yC_1\right)^2\right. \\ & \left. + \left(RC_2 - \mu_xC_1 + \mu_yS_1\right)^2 - \left(\mu_xS_1 - \mu_yC_1\right)^2\right\}\end{aligned}$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos n\theta \exp\left(-f(R,\theta)/2q^2\right) d\theta \qquad n = 0,1,2 \; ,$$

$$S_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin n\theta \exp\left(-f(R,\theta)/2q^2\right) d\theta \qquad n = 1,2 \; ,$$

$$q^2 = \sigma_x^2\sigma_y^2 - \sigma_{xy}^2 \; ,$$

and

$$f(R,\theta) = \sigma_y^2 \left(R \cos\theta - \mu_x\right)^2 - 2\sigma_{xy} \left(R \cos\theta - \mu_x\right)\left(R \sin\theta - \mu_y\right) + \sigma_x^2 \left(R \sin\theta - \mu_y\right)^2 \; .$$

They also give a series approximation for the variance of $\hat{R}$.

If the probability density function of the weapon impact-points is circular Gaussian and centered on the target, the estimation of the radius R corresponding to a given destruction probability P is much simplified. Usually, one sets P equal to 0.5; weapons analysts will recognize this as the problem of estimating the circular probable error, abbreviated CEP. It is easy to show that the radius R is in this case equal to $\left(2 \log_e 2\right)^{1/2}\sigma = 1.1774\sigma$; therefore, the problem of estimating R is equivalent to the problem of estimating σ from a sample $\left(x_1,y_1\right), \left(x_2,y_2\right), \ldots, \left(x_n,y_n\right)$ from the circular Gaussian distribution with zero mean.

Moranda (1959) derives the minimum-variance unbiased estimate of σ:

$$\hat{\sigma} = \frac{\Gamma(n)}{\Gamma(n+1/2)} \left\{ \sum_{i=1}^{n} \left(x_i^2 + y_i^2\right)/2 \right\}^{1/2} \; .$$

This estimate is unbiased, and its variance is

$$\mathrm{var}(\hat{\sigma}) = \left\{ \frac{n\Gamma^2(n)}{\Gamma^2(n+1/2)} - 1 \right\} \sigma^2 \; .$$

Consequently the variance of $\hat{R}$ is $(1.1774)^2 \, \mathrm{var}(\hat{\sigma})$. As n approaches infinity, the quantity in braces approaches $1/4n$.

Moranda (1959), Kamat (1962) and Inselmann and Granville (1967) have proposed a variety of alternative estimates of σ, in the hope that simplicity of calculation will compensate for increase of variance. Note that all of these estimates assume that the circular Gaussian distribution of errors has a mean at (0,0).

1. Mean Radial Error. Calculate $\sum_{i=1}^{n} \left(x_i^2 + y_i^2\right)^{1/2} \Big/ n = r$. This statistic has an expected value $E(r) = (\pi/2)^{1/2}\sigma$; therefore, an unbiased estimate of σ is given by $(2/\pi)^{1/2} r$. The variance of $(2/\pi)^{1/2} r$ is $\left((4/\pi) - 1\right)\sigma^2/n$, which is 1.10 times as large as the variance of the best estimate $\hat{\sigma}$.

2. Mean Deviation. Calculate $\sum_{i=1}^{n} \left(|x_i| + |y_i|\right)/2n = d$. This statistic has an expected value $E(d) = (2/\pi)^{1/2}\sigma$; therefore, an unbiased estimate of σ is given by $(\pi/2)^{1/2} d$. The variance of $(\pi/2)^{1/2} d$ is $(\pi-2)\sigma^2/4n$, which is 1.14 times as large as the variance of the best estimate $\hat{\sigma}$.

3. Radial Order Statistic. Arrange the quantities $r_i = \left(x_i^2 + y_i^2\right)^{1/2}$ in order from smallest to largest: $r_{(1)} \le r_{(2)} \le \ldots \le r_{(n)}$. Let $r_{(j)}$ denote the jth order statistic, and associate with j the quantity $p = j/(n+1)$. If j is chosen so that p is equal to 0.797, then the statistic

$$s = r_{(j)} \left(2\left(-\log_e(1-p)\right)\right)^{-1/2} = 0.558\, r_{(j)}$$

is an unbiased estimate of σ and has a variance equal to $0.396\,\sigma^2/n$, which is 1.58 times as large as the variance of the best estimate $\hat{\sigma}$. (In practice, since j and n must be integers, one can only approximate p by p'; the multiplicative factor becomes $\left(2\left(-\log_e(1-p')\right)\right)^{-1/2}$).

The following argument shows why $p = 0.797$ was chosen. It is a well-known result from classical statistics that, for large n, the jth order statistic in a sample of size n from a probability density function f(x) is approximately distributed as a Gaussian random variable with

$$E\left(r_{(j)}\right) = x_p$$

$$\mathrm{var}\left(r_{(j)}\right) = p(1-p)/n\left(f\left(x_p\right)\right)^2$$

where

$$\int_{-\infty}^{x_p} f(x)\,dx = p \quad .$$

In this application, f(x) is equal to $\left(x/\sigma^2\right)\exp\left(-x^2/2\sigma^2\right)$; therefore, $x_p/\sigma = \left(2\left(-\log_e(1-p)\right)\right)^{1/2}$ and $f\left(x_p\right) = (1-p)\left(x_p/\sigma^2\right)$. The statistic $r_{(j)}\left(2\left(-\log_e(1-p)\right)\right)^{-1/2}$ is an unbiased estimate of σ with variance $p(1-p)^{-1}\left(2\left(-\log_e(1-p)\right)\right)^{-2}\sigma^2/n$. It is a straightforward matter to ascertain that the value of p which minimizes this variance is the solution to the equation $-\log_e(1-p) = 2p$, or $p = 0.797$. In other words, among all unbiased estimates of σ using a single order statistic, this has the smallest possible variance. Sometimes the median (divided by 1.1774) is used as an estimate of σ; however, its variance is $0.521\,\sigma^2/n$.

Strictly speaking, the above estimates $\hat{\sigma}$, r, d and s should be used only when one is certain that the probability distribution function of impact points is circular Gaussian with a mean at (0,0). If these assumptions are not true, the estimate of σ can be seriously in error. To protect oneself against such a possibility, one can use a more "robust" estimate protecting against errors in the assumptions, at the price of increased variability of the estimate if the assumptions are, in fact, correct.

Suppose, for example, that the mean of the distribution of impact points is not located at (0,0). One can perform statistical tests on the sample values $\left(x_1,y_1\right),\ldots,\left(x_n,y_n\right)$ to determine whether or not this hypothesis is reasonable. Alternatively, one can construct estimates insensitive to this failure by replacing x_i and y_i with $x_i - \bar{x}$ and $y_i - \bar{y}$, respectively, where

$$\bar{x} = \sum_{i=1}^{n} x_i/n \quad \text{and} \quad \bar{y} = \sum_{i=1}^{n} y_i/n \quad .$$

Moranda (1959) suggests the following analogue to $\hat{\sigma}$:

$$\hat{\sigma}' = \frac{\Gamma(n-1)}{\Gamma(n-1/2)} \left\{ \sum_{i=1}^{n} \left((x_i-\bar{x})^2 + (y_i-\bar{y})^2 \right) / 2 \right\}^{1/2}$$

This estimate is unbiased, and its variance is

$$\mathrm{var}(\hat{\sigma}') = \left\{ \frac{(n-1)\,\Gamma^2(n-1)}{\Gamma^2(n-1/2)} - 1 \right\} \sigma^2 .$$

As n approaches infinity, the bracketed quantity approaches $1/4(n-1)$. When $n = 5$, 10 and 20, the quantity $\mathrm{var}(\hat{\sigma}')/\mathrm{var}(\hat{\sigma}) = 1.26$, 1.11 and 1.06 respectively. The price paid for robustness is quite small.

Kamat (1962), Grubbs (1964b) and Cacoullos and DeCicco (1967) have proposed a variety of alternative estimates of σ, in the hope that simplicity of calculation will compensate for increase of variance.

1. Mean Radial Error. Calculate $\sum_{i=1}^{n} \left((x_i-\bar{x})^2 + (y_i-\bar{y})^2 \right)^{1/2} / n = r'$.

This statistic has an expected value $E(r') = ((n-1)\pi/2n)^{1/2}\sigma$; therefore, an unbiased estimate of σ is given by $(2n/(n-1)\pi)^{1/2} r'$. The variance of this estimate is given in Grubbs (1964b). When $n = 5$, 10 and 20, the quantity $\mathrm{var}(r'/E(r'))/\mathrm{var}(r/E(r)) = 1.23$, 1.11 and 1.06 respectively. Again, the price is small.

2. Mean Deviation. Calculate $\sum_{i=1}^{n} \left(|x_i-\bar{x}| + |y_i-\bar{y}| \right) / 2n = d'$. This statistic has an expected value $E(d') = (2(n-1)/\pi n)^{1/2}\sigma$; therefore, an unbiased estimate of σ is given by $(\pi n/2(n-1))^{1/2} d'$. The variance of this estimate is given in Kamat (1962). When $n = 5$, 10 and 20, the quantity $\mathrm{var}(d'/E(d'))/\mathrm{var}(d/E(d)) = 1.22$, 1.09 and 1.05 respectively. Again, the price is small.

3. Radial Order Statistic. The analogue to s has not been derived. A variety of estimates of σ based on the extreme order statistics in the sample have been proposed. Although such estimates have the virtue of not depending upon the location of the mean of the impact-point distribution, their variance increases rapidly with n; consequently they should be considered only for quite small sizes (say, 20 or less). In one dimension, the range $w = x_{(n)} - x_{(1)}$ is a natural

statistic to use for estimating σ. The expected value of w is $d_n\sigma$, and the factor d_n is tabulated for $2 \le n \le 20$ in Table 20 of Pearson and Hartley (1954). When one goes to two dimensions, various generalizations of the range have been proposed by Kamat (1962), Grubbs (1964b), Daniels (1952) and Bradley (1965):

$$w_1 = \frac{1}{2}\left(x_{(n)} - x_{(1)} + y_{(n)} - y_{(1)}\right) \qquad \text{(average range)}$$

$$w_2 = \left\{\left(x_{(n)}-x_{(1)}\right)^2 + \left(y_{(n)}-y_{(1)}\right)^2\right\}^{1/2} \qquad \text{(diagonal range)}$$

$$w_3 = \max_{i,j} \left\{\left(x_i-x_j\right)^2 + \left(y_i-y_j\right)^2\right\}^{1/2} \qquad \text{(extreme spread)}$$

w_4 = radius of smallest circle covering $\left(x_1,y_1\right),\ldots,\left(x_n,y_n\right)$

w_5 = perimeter of smallest convex polygon containing $\left(x_1,y_1\right),\ldots,\left(x_n,y_n\right)$.

The variability of the first four estimates (after corrections for bias so that $E\left(b_i w_i\right) = \sigma$) is quite similar; for example, if n = 10, $\mathrm{var}\left(b_1w_1\right) = 0.0334\ \sigma^2$, $\mathrm{var}\left(b_2w_2\right) = 0.0331\ \sigma^2$, $\mathrm{var}\left(b_3w_3\right) = 0.0376\ \sigma^2$ and $\mathrm{var}\left(b_4w_4\right) = 0.0365\ \sigma^2$. The statistical behavior of w_5 is not known. The statistic w_1 is the simplest to calculate. Its bias correction is $b_1 = 1/d_n$; $\mathrm{var}\left(w_1/d_n\right) = V_n\sigma^2/2d_n^2$, where d_n and V_n are tabulated in Table 20 of Pearson and Hartley (1954). For n = 5, 10 and 20, the quantity $\mathrm{var}\left(w_1/d_n\right)/\mathrm{var}(\hat{\sigma}') = 1.07,\ 1.19$ and 1.43 respectively.

Kamat (1962) considers yet another assumption failure. If one is worried about a slow "slippage" of the mean of the impact-point distribution as the observations $\left(x_i,y_i\right)$ are being collected, he suggests the statistic

$$d'' = \sum_{i=1}^{n-1} \left(|x_i-x_{i+1}| + |y_i-y_{i+1}|\right)/2(n-1) \quad .$$

The expected value of d'' is $2/\sqrt{\pi}$, and the variance is approximated by

$$\mathrm{var}(\sqrt{\pi}\ d''/2) = \frac{\pi}{8}\left(\frac{1.052}{n-1} - \frac{0.326}{(n-1)^2}\right)\sigma^2 \ .$$

If one uses this estimate, one pays a somewhat larger price in increased variability when the impact-point probability density function is centered on the target: as n goes to infinity, the ratio of this variance to that of the minimum-variance estimate is 1.68. However, no other estimate presented above protects against a slow slippage of the mean.

2.6.3 Estimation of the Parameters of a Diffused Gaussian Damage Function

So far, all estimation problems have been carried out in the context of a cookie-cutter damage function — the target is assumed to be destroyed if and only if the weapon lands within R of it. In such a situation, the probability that a weapon will land inside a circle of radius R is of primary importance.

Suppose, however, that one has a diffused Gaussian damage function $\exp\left(-r^2/2b^2\right)$ instead. Assume that a point target is located at (0,0) and that the impact points of weapons directed at the target have a circular Gaussian probability density function centered on the target and with variance σ^2. Under these circumstances, the probability of target destruction by a single weapon is given by $P = b^2/\left(\sigma^2+b^2\right)$.

Assume that one has fired n weapons against the target, m of which destroyed the target. For those m weapons, the miss-distances $r_1, r_2, \ldots, r_m$ are known; for the others, the miss-distances are unknown. (This situation might occur if each weapon was set off by a proximity fuse, which operated at distance r with probability $\exp\left(-r^2/2b^2\right)$.)

Obviously, one can estimate P by $\hat{P} = m/n$. However, if one is interested in estimating σ^2 and b^2, the components of P, more sophisticated estimates are required. Thompson (1958) derives the maximum-likelihood estimates $\hat{\sigma}^2$ and $\hat{b}^2$. To do this, he forms the likelihood function (the probability of getting m target destructions out of n weapons with miss-distances $r_1, r_2, \ldots, r_m$):

$$L\left(b^2,\sigma^2\right) = \binom{n}{m}\left(\frac{\sigma^2}{b^2+\sigma^2}\right)^{m-n}\left(\frac{b^2}{b^2+\sigma^2}\right)^{n}\exp\left(-Q\left(\frac{1}{b^2}+\frac{1}{\sigma^2}\right)\right)\prod_{i=1}^{m} r_i \ ,$$

where

$$Q = \frac{1}{2}\sum_{i=1}^{m} r_i^2 \quad .$$

Differentiating $L(b^2,\sigma^2)$ with respect to b^2 and σ^2 and setting these expressions equal to zero, one obtains two equations to solve for σ^2 and b^2. These are known as the maximum-likelihood estimates of σ^2 and b^2:

$$\hat{\sigma}^2 = nQ/m^2 \quad , \qquad \hat{b}^2 = nQ/m(n-m) \quad .$$

It is comforting to note that $\hat{b}^2/(\hat{\sigma}^2+\hat{b}^2) = m/n$, the estimate proposed earlier for P.

2.7 OFFENSIVE SHOOT-ADJUST-SHOOT STRATEGIES

The preceding sections of this chapter have all considered situations in which a salvo of weapons has been simultaneously directed against an undefended point target. Suppose, however, that the offense can observe the impact-points of his first (i-1) weapons before directing his ith weapon at the target. He may discover, for example, that his weapons are clustering about an offset aiming-point rather than the target. Obviously, he can compensate for this offset by directing future weapons not at the target but at an imaginary offset aiming-point on the opposite side of the target. The procedure outlined in this section calculates the optimum compensation to be applied to each weapon, based on a knowledge of the earlier impact-points and their associated compensations.

Consider the following one-dimensional bombing problem. A point target is located at the origin; it is destroyed if and only if a weapon impacts between -R and +R. Assume that the weapon impact-point z_i is the sum of two components: a fixed bias B of unknown magnitude, and a random aiming-error x_i having a Gaussian probability density function with a variance of σ^2. A non-lethal weapon is directed at the target for calibration purposes; its impact-point is $z_0 = B + x_0$. If one aims the first lethal weapon at $-z_0$, then the new bias is equal to $B - z_0 = -x_0$, and the impact-point, $z_1 = x_1 - x_0$, will have a Gaussian probability density function with mean centered on the target and variance $2\sigma^2$. Note that the calibration shot has eliminated the unknown bias B; the probability density function of the impact-point of the first lethal weapon is completely known.

What bias should be given to the second lethal weapon? The probability of kill of the second weapon will be maximized if the variance of z_2 is minimized. It is not difficult to show that this is accomplished if the bias is changed by $-(1/2)z_1$; the new bias is equal to $-(1/2)\left(x_0+x_1\right)$ and the impact-point, $z_2 = x_2 - (1/2)\left(x_0+x_1\right)$, will have a Gaussian probability density function with variance $(3/2)\sigma^2$. If one applies the same argument to the third weapon, the bias is changed by $-(1/3)z_2$, and the impact-point, $z_3 = x_3 - (1/3)\left(x_0+x_1+x_2\right)$ has a variance of $(4/3)\sigma^2$. The general bias-correction procedure is now evident; as n approaches infinity, the impact-point will have a Gaussian probability density function with variance σ^2 centered on the target. Nadler and Eilbott (1971) show that this bias-correction procedure is optimum (in the sense of minimum variance at each stage) among all bias-correction procedures which are linear functions of the impact-point observations z_i.

For an attack of n weapons, Nadler and Eilbott derived the probability that the target will be destroyed if the above bias-correction procedure is independently applied at each stage to the x-component and the y-component of the two-dimensional impact-point error:

$$P = 1 - \exp\left(-\left(R^2/2\sigma^2\right)\sum_{i=1}^{n} i/(i+1)\right) .$$

In the two-dimensional situation, the aiming-error of each weapon is assumed to have a circular Gaussian probability density function with variance σ^2; the target is destroyed if any of the n weapons impact within a distance R. Nadler and Eilbott prove that the two-dimensional bias-correction procedure is optimum, not only in the sense of minimizing the variance of the x-component and the y-component of the impact-point error at each stage, but also in the sense of maximizing the probability of target kill with n weapons assuming a cookie-cutter damage function of radius R.

The summation in the exponent is equal to $n+1 - \gamma - \psi(n+2)$, where γ is Euler's constant and $\psi(n)$ is the digamma function (which is tabulated). The summation can be approximated by the expression $n - \log_e(2n+3)/3$.

However, this bias-correction procedure should not be applied in all circumstances. For example, if B is known to be zero (or small with respect to the standard deviation σ), then the attacker will obviously do better if he leaves the bias alone. In fact, the probability P that the target will be destroyed by an attack of n weapons increases to

$$P = 1 - \exp\left(-nR^2/2\sigma^2\right)$$

if no bias exists and no bias-correction is used. How should one decide when to use the procedure? In order to eliminate the variable n, it is convenient to introduce a new criterion of effectiveness: the expected number of attacking weapons required to destroy the target. If the bias B is less than the critical bias B* given in the table below, then the expected number of attacking weapons needed to destroy the target if no bias-correction is used is less than the expected number of attacking weapons needed to destroy the target if the optimum bias-correction is used; if the bias B is greater than the critical bias B*, then the opposite situation is true. For the details of the numerical calculations leading to this table, see Nadler and Eilbott (1971).

TABLE OF CRITICAL INITIAL BIAS LEVELS

R/σ	0	0.1	0.2	0.3	0.4	0.5	0.6
B^*/σ	0	0.21	0.35	0.46	0.55	0.63	0.69

However, the attacker will not be able to specify the bias B with any degree of precision. Therefore, it is necessary to introduce the following approximate rule: use the bias-correction procedure if the probability that $B \geq B^*$ exceeds 1/2, and do not use the bias-correction procedure if this probability is less than 1/2.

It is important to distinguish the shoot-adjust-shoot strategy presented in this section from the more well-known shoot-look-shoot strategy discussed in the literature. The objective of a shoot-look-strategy is to conserve weapons by observing after each firing whether or not the target has been destroyed. By contrast, the shoot-adjust-shoot strategy attempts to maximize the probability of target destruction, given that n weapons have been made available for this task. When one is using a cookie-cutter damage function, it is appropriate to combine a shoot-adjust-shoot and a shoot-look-shoot strategy. The information needed to carry out the bias-correction (that is, the distance the last weapon impacted from the target) is exactly the information required to determine whether or not the target has been destroyed (the distance is compared with the lethal radius R). Nadler and Eilbott proved that the bias-correction procedure is optimum when shoot-adjust-shoot and shoot-look-shoot are combined. Note that this problem is non-trivial; the probability of target kill by the nth weapon must be conditioned on the fact that the first (n-1) weapons all impact outside a circle with radius R centered on the target.

For other damage functions (for example, the diffused Gaussian damage function), it is more natural to consider shoot-adjust-shoot and shoot-look-shoot strategies separately. The observation of a weapon impact-point does not immediately tell the offense whether or not the target has been destroyed; the condition of the target must also be observed. It can be shown that the above bias-correction procedure is optimum for any circularly symmetric damage function. The probability of target destruction using a diffused Gaussian damage-function is

$$P = 1 - \prod_{i=1}^{n} (i+1)\sigma^2 / \left(ib^2 + (i+1)\sigma^2\right) .$$

If one attacks area targets instead of point targets using a shoot-adjust-shoot strategy, the bias-correction problem appears to be considerably more difficult. If the first weapon impacts below the target, for example, it may be desirable to overcompensate for the apparent bias B in order to place the second weapon above the target and minimize the chance of overlap. The bias-correction soon becomes a complicated function of the geometry of all impact-points (not simply the radial distance between the last impact-point and the target, as for a point target).

2.8 ATTACK EVALUATION BY DEFENSE USING RADAR INFORMATION

In Sections 2.1.1-2.5.4, various formulas have been presented for calculating target damage under a variety of assumptions about the nature of the target and the damage function. In all these models, it is assumed that the impact-point probability density function, $p(x,y)$, of the attacking weapon is bivariate Gaussian. Suppose, however, that the defense is able to estimate the impact-point of the weapon by means of radar observations of the weapons in flight. How does this additional information modify the expectation of target damage? In short, can one determine the conditional threat posed to a target by an incoming weapon? In many of the models of active defense considered in future chapters, this sort of information could be of considerable value.

This problem has been analyzed in some detail. Specifically, consider two kinds of damage functions (cookie-cutter with radius R, diffused Gaussian with standard deviation b) and three kinds of targets (point, u[illegible]-valued circular with radius K, and Gaussian with standard deviation σ_T). The target is centered on the origin of a two-dimensional coordinate system. Prior to the radar observation, the probability density function of the weapon impact-point is assumed to be a bivariate Gaussian with mean $\left(x_0, y_0\right)$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}.$$

Using the radar, the defense estimates that the weapon will impact at the point $\left(x_p, y_p\right)$. The true impact-point of the weapon is assumed to have a bivariate Gaussian probability density function with mean $\left(x_p, y_p\right)$ and covariance matrix

$$\overline{\Sigma} = \begin{pmatrix} \overline{\sigma}_x^2 & \overline{\sigma}_{xy} \\ \overline{\sigma}_{xy} & \overline{\sigma}_y^2 \end{pmatrix}.$$

Let $\left(x_p, y_p\right)$ be denoted by the column vector $\underline{Z}$. If $\overline{\Sigma}$ is large with respect to Σ, then the formulas presented in Sections 2.1.1-2.5.4 can be used to evaluate potential target damage. In the earlier work, the covariance term could be eliminated by an appropriate rotation of the coordinate system; however, this is no longer possible because the error ellipsoids of weapon impact-point and radar prediction may not be parallel to each other.

If one assumes a diffused Gaussian damage function, one can write down in closed analytic form the probability of kill for a point target or the expected fraction of value destroyed for a Gaussian target. For the point target, the probability of kill is

$$P = b^2 \left|\beta \underline{I} + \underline{D}\right|^{-1/2} \exp\left(-\frac{1}{2} \underline{C}^T (\beta \underline{I} + \underline{D})^{-1} \underline{C}\right)$$

where $\underline{I}$ is the two-by-two identity matrix, $\underline{D}$ is the two-by-two matrix $\Sigma(\Sigma + \overline{\Sigma})^{-1}\overline{\Sigma}$, $\underline{C}$ is the column vector $\Sigma(\Sigma + \overline{\Sigma})^{-1}\underline{Z}$, and β is a constant equal to b^2. E(f), the expected fraction of value destroyed for a Gaussian target, can be expressed by the same equation, provided that β is changed to $\beta' = b^2 + \sigma_T^2$.

If one assumes a cookie-cutter damage function, the probability of kill for a point target or the expected fraction of value destroyed for a Gaussian target is obtained by integrating a circular Gaussian probability density function over an offset ellipse. This can be accomplished using the tables discussed in Section 2.1.4; however, the specification of the ellipse is somewhat involved and is omitted here.

The evaluation of the expected fraction destroyed of a uniform-valued circular target is considerably more difficult. If a diffused

Gaussian damage function is used, the integration of a circular Gaussian probability density function over an offset ellipse is again necessary.

2.9 SUMMARY

This chapter presents a wide variety of formulas to calculate target damage by a salvo of identical weapons with a common aim-point and independent aiming-errors. If the target is small with respect to the cookie-cutter damage radius, the probability of survival for offset ellipsoid aiming-errors can be calculated; however, little is known about the optimum aim-point to use against clusters of point targets. If the target is large with respect to the damage-radius, the expected damage by a salvo of weapons becomes much more difficult to calculate; for simplicity, two idealized distributions of target value — the uniform-valued circular target and the Gaussian-valued target — are introduced. However, it is possible to obtain computable approximations for the expected damage by an offset ellipsoid attack only when the cookie-cutter damage function is replaced by a diffused Gaussian damage function.

If the attacker can control the relative locations of the aim-points of different weapons, he can optimize the expected target damage. Such precise control is rarely possible, however, and the attacker may only be able to control the standard deviation of weapons about a single aim-point, or (better still) the probability density function of weapons about a single aim-point.

The evaluation of the above target damage formulas requires a knowledge of quantities such as the probability that a random weapon will land inside a circle of specified size; these quantities can be estimated from observed weapon impact-points using a variety of techniques from applied statistics. If the attacker can allocate weapons sequentially instead of in a salvo, he may be able to correct biases in his aim by observing the impact-points of his weapons with respect to the target. The final section anticipates succeeding chapters by showing how the defense can calculate impact-point probability density functions conditioned by observations on the incoming weapon, thus sorting out the more threatening from the less threatening.

CHAPTER THREE

DEFENSE OF A TARGET OF UNSPECIFIED STRUCTURE

This chapter is the first one in this monograph to consider problems of active defense by a stockpile of defensive missiles. The simplest such problems are those in which it is possible to ignore the structure of the targets defended. For example, suppose that one is interested in defending a point target with a stockpile of defensive missiles against a group of offensive weapons. When one is dealing with a single target, the natural criterion of effectiveness to use is the probability that the target survives. As another example, suppose that the target has an area of uniform value which is so large that it can absorb the entire offensive stockpile without any overlap of damage regions. In this case, the damage will be proportional to the number of weapons that penetrate, and a natural effectiveness criterion is the expected number of penetrators.

The fact that the target structure is ignored in these criteria usually (but not always) simplifies the analysis. Consequently, the models of this chapter are often used to approximate more complicated situations or are used as components of more elaborate models.

The following notation is used throughout this chapter: the number of weapons in the defensive and offensive stockpiles are designated by m and n, respectively. If the target is a point target, it is generally assumed that an unintercepted offensive weapon destroys the target if and only if it lands within a distance R of the target. All offensive weapons have impact-point errors drawn from the same probability density function; furthermore, the impact-point errors are independent from one weapon to the next. It is also assumed that defensive missiles have a reliability ρ — that is, if a defensive missile is assigned to an offensive weapon, the latter is destroyed with probability ρ. Furthermore, the individual missiles of a defensive salvo operate independently of each other — if the salvo is of size i, the probability of offensive weapon destruction is equal to $1 - (1-\rho)^i$. Finally, it is sometimes assumed that an unintercepted offensive weapon has a fixed probability p of success against the target.

Ordinarily, the reliability ρ is assumed to be known to the defense. Suppose that the defense can observe where all n of the offensive weapons will land before committing any defensive missiles, and suppose that the defense also knows the lethal radius R. Then the optimum defense strategy is trivial — he observes that

$n' \leq n$ of the offensive weapons will land within a distance R, and salvos his m defensive missiles as evenly as possible against each of the offensive weapons. Given that $n' \leq m$ offensive weapons land within a distance R, the probability of target destruction is

$$P_{n'} = 1 - \left(1-(1-\rho)^k\right)^{n'-r} \left(1-(1-\rho)^{k+1}\right)^r \quad ,$$

where

$$k = [m/n'] \qquad \text{and} \qquad r = m - n'k \quad .$$

If $r = 0$, the formula simplifies to

$$P_{n'} = 1 - \left(1-(1-\rho)^k\right)^{n'} \quad .$$

If $n' > m$, $P_{n'}$ is equal to unity.

Denote the probability that an offensive weapon lands within R of the target by P. Then the unconditional probability of target destruction prior to the attack is

$$P = \sum_{i=0}^{n} \binom{n}{i} P^i (1-P)^{n-i} P_i \quad .$$

The quantity P was calculated in Chapter 2 under a wide variety of different assumptions about the impact-point distribution. For example, if the probability density function of the impact-point distribution is circular Gaussian centered on the target and with standard deviation $\sigma_x = \sigma_y = \sigma$, then

$$P = 1 - \exp(-R^2/2\sigma^2) \quad .$$

The rest of this chapter derives optimum defense strategies for various modifications of this standard defense problem. On the one hand, the defense may not be able to see the attack before committing its missiles, or may not know the lethal radius R; on the other hand, the defense may be able to use shoot-look-shoot — that is, to determine whether or not a defensive missile has destroyed an offensive weapon before directing a second missile against the same weapon.

3.1 DEFENSE STRATEGIES AGAINST WEAPONS OF UNKNOWN LETHAL RADIUS

Suppose that the defense can see the entire attack before launching a single defensive missile; that is, he can predict that the offensive weapons will impact at distances $r_1 \le r_2 \le \dots \le r_n$ from the target. He is willing to assume that all n weapons have the same lethal radius R, but he does not know what this is. Intuitively, one would like to allocate more defensive missiles to those weapons with small values of r_i than to those with large r_i.

It is difficult to allocate defensive missiles to minimize the probability of target destruction when R is unknown. A procedure which achieves minimum probability for one (hypothesized) value of R will not work for another value of R. However, the following criterion should work reasonably well over a range of R: allocate the defensive missiles to maximize the expected distance to the nearest penetrator (that weapon against which all defensive missiles allocated fail). This is an example of a robust strategy as discussed in Section 1.2. Let m_i denote the number of defensive missiles allocated to the ith closest weapon, and let A_i denote the probability that the ith closest weapon will penetrate: $A_i = (1-\rho)^{m_i}$. Then the expected distance to the nearest penetrator is

$$E = r_1A_1 + r_2A_2(1-A_1) + \dots + r_jA_j \prod_{i=1}^{j-1} (1-A_i) + r_{j+1} \prod_{i=1}^{j} (1-A_i) \quad ,$$

if the nearest j offensive weapons are assigned defensive missiles:

$$\sum_{i=1}^{j} m_i = m \ , \quad m_i > 0 \qquad \text{for} \qquad 1 \le i \le j \ , \quad m_{j+1} = 0 \quad .$$

The problem of maximizing E subject to the constraint $\prod_{i=1}^{j} A_i = A = (1-\rho)^m$ can be solved by dynamic programming. However, it can be solved approximately by allowing the unknowns to become continuous and treating A as an unknown. This approach may be reasonable when ρ is small. To proceed, substitute $A(A_1A_2 \dots A_{j-1})^{-1}$ for A_j in the above equation and solve the system of equations $\partial E / \partial m_i = 0$, $i = 1,2,\dots,j-1$ to obtain the optimum values of $A_1, A_2, \ldots, A_{j-1}$. A can be chosen arbitrarily; select that value of A that makes $A_j = 1$ (that is, let $A = A_1 A_2 \ldots A_{j-1}$). By this procedure one arrives, after simplification, at the following equations for the optimal A_k^*:

$$-r_k A_k^* + \sum_{i=k+1}^{j-i} r_i A_k^* A_i^* \prod_{t=k+1}^{i-1} \left(1-A_t^*\right)$$

$$+ r_j \prod_{t=k+1}^{j-1} \left(1-A_t^*\right) - r_{j+1} \prod_{t=k}^{j-1} \left(1-A_t^*\right) = 0 \;, \quad k = 1,2,\ldots,j-1 \quad .$$

One can derive a recurrence for the A_k^* in terms of the r_k. Solving the kth equation above for A_k^*, one has

$$A_k^* = \left(r_{j+1} - r_j\right) \prod_{t=k+1}^{j-1} \left(1-A_t^*\right) \Bigg/ \left(\sum_{i=k+1}^{j-1} r_i A_i^* \prod_{t=k+1}^{i-1} \left(1-A_t^*\right) + r_{j+1} \prod_{t=k+1}^{j-1} \left(1-A_t^*\right) - r_k \right) .$$

If one takes P_k and Q_k as the numerator and denominator, respectively, of the above fraction, one arrives at the following recursion:

$$P_k = P_{k+1}\left(1-A_{k+1}^*\right) ,$$

$$Q_k = Q_{k+1}\left(1-A_{k+1}^*\right) + r_{k+1} - r_k ,$$

$$A_k^* = P_k / Q_k ,$$

for $k = j-2, j-3, \ldots, 1$, where $P_1 = r_{j+1} - r_j$, $Q_1 = r_{j+1} - r_{j-1}$, and $A_1^* = P_1/Q_1$. This recurrence enables one to calculate successively $A_{j-1}^*, A_{j-2}^*, \ldots, A_1^*$, and hence the corresponding $m_i^* = \log A_i^* / \log(1-\rho)$.

It is of interest to give explicit formulas for the first few A_k^*. This task is somewhat simplified if one sets $R_k = P_k Z_k$, $S_k = Q_k Z_k$, where $Z_k = \prod_{i=k+1}^{j-1} Q_i$. This leads to another recurrence for the A_k^*:

$$Z_k = Z_{k+1} \cdot S_{k+1} ,$$
$$R_k = R_{k+1}\left(S_{k+1} - R_{k+1}\right) ,$$
$$S_k = S_{k+1}\left(S_{k+1} - R_{k+1}\right) + Z_k\left(r_{k+1} - r_k\right) ,$$
$$A_k^* = R_k / S_k ,$$

for $k = j-2, j-3, \ldots, 1$, where $Z_1 = 1$, $R_1 = r_{j+1} - r_j$, $S_1 = r_{j+1} - r_{j-1}$, $A_1^* = R_1 / S_1$. The advantage of this apparently more cumbersome recurrence is the avoidance of fractions in the calculation of Z_k, R_k, and S_k. The expressions for the first three A_k^* are as follows:

$$A_{j-1}^* = \left(r_{j+1} - r_j\right) / \left(r_{j+1} - r_{j-1}\right) ,$$
$$A_{j-2}^* = A_{j-1}\left(r_j - r_{j-1}\right) / \left(r_j - r_{j-2}\right) ,$$
$$A_{j-3}^* = A_{j-2}\left(\left(r_j - r_{j-2}\right)\left(r_{j+1} - r_{j-1}\right) - \left(r_j - r_{j-1}\right)\left(r_{j+1} - r_j\right)\right) / \left(\left(r_j - r_{j-3}\right)\left(r_{j+1} - r_{j-1}\right) - \left(r_j - r_{j-1}\right)\left(r_{j+1} - r_j\right)\right) .$$

The optimum continuous allocation of defensive resources has been presented above for certain discrete values of A — namely, those corresponding to each possible choice of j. If one has defensive resources associated with an A between two of these discrete values, one can perform linear interpolation on the missile allocations m_i corresponding to the A_i, in order to produce an approximate solution.

A numerical illustration may be helpful here. Suppose $\rho = 0.4$ and $m = 7$, and suppose that offensive weapons are observed to have miss distances of 1, 4, 5, 7 and 8 from the target. For $j = 2$, $A_1^* = 1/4$ and the corresponding m_1 is 2.71. For $j = 3$, $A_1^* = 1/6$, $A_2^* = 2/3$ and the corresponding m_1 and m_2 are 3.52 and 0.80. For $j = 4$, $A_1^* = 7/72$, $A_2^* = 2/9$, $A_3^* = 1/3$ and the corresponding m_1, m_2 and m_3 are 4.56, 2.94 and 2.15. The defensive stockpile of 7 is located .503 of the way between 4.32 and 9.65; therefore $m_1 = 3.52 + (.503)(4.56-3.52) = 4.05$,

$m_2 = 0.80 + (.503)(2.94-0.80) = 1.87$, and $m_3 = (.503)(2.15) = 1.08$. Of course, since it is impossible to allocate fractional missiles, a good practical allocation is $m_1 = 4$, $m_2 = 2$, $m_3 = 1$. The best way of rounding the continuous variables will not always be so obvious.

Once the A^*_{j-i} are known, the expected distance to the nearest penetrator, E, can be readily calculated. However, explicit algebraic expressions for E are cumbersome. For j = 2,

$$E = \left(r_1 r_3 - r_2^2\right) / \left(r_3 - r_1\right) \leq r_2 \quad .$$

One practical difficulty in using this allocation scheme is that the A^*_{j-i} must be computed during the course of the engagement rather than beforehand. As an alternative to the above procedure, one can derive allocation strategies corresponding to a different criterion: allocate the defensive missiles to maximize the expected rank of the nearest offensive weapon penetrating the defense. In other words, actual distances r_i are replaced with the ranks i and the problem solved as before. Now, it is possible to store an allocation strategy in a computer before the actual engagement occurs; it will be a function of $A = (1-\rho)^m$ alone. In general, this should lead to allocations very similar to those produced by the above method.

The problem of allocating defensive missiles to maximize the rank of the nearest penetrator is equivalent to the problem of allocating defensive missiles to maximize the expected number of weapons to the first penetrator in an attack of indeterminate size. This latter problem is discussed in Sections 3.2.1 and 3.2.2.

3.2 DEFENSE STRATEGIES AGAINST A SEQUENTIAL ATTACK OF UNKNOWN SIZE

Suppose that the defense knows the lethal radius, R, of an offensive weapon but does not know the size of the offensive attack. In particular, assume that offensive weapons appear one at a time, and the defense must decide how many missiles to allocate to each one before the next weapon appears. Clearly, the defense will only allocate missiles to that subset of the attack which has predicted impact-points within a distance R of the target. Intuitively, it is clear that one would like to allocate more defensive missiles to earlier weapons and fewer to later ones, in order to avoid target destruction while one still has a substantial stockpile of defensive missiles.

3.2.1 Maximizing the Expected Rank of the First Penetrator

It is difficult to allocate defensive missiles to minimize the probability of target destruction when the attack size is unknown. A procedure that achieves minimum probability for one (hypothesized) attack size will not work for another value. However, the following criterion should work reasonably well over a range of attack sizes: allocate the defensive missiles to maximize the expected number of weapons to the first penetrator (that weapon against which all defensive weapons allocated fail). This approach may be useful in another situation. Often a defense objective is stated, somewhat vaguely, as that of maximizing the price (in number of weapons) the offense must pay to achieve a high confidence of target destruction. Rather than debate the somewhat arbitrary level of confidence to be set, it may be plausible to modify such a criterion to that of maximizing the expected rank of the first penetrator. This procedure will tend to have the effect of charging a high price over a range of levels of confidence.

Even when the attack size is known, a strategy of maximizing the expected rank of the first penetrator may make sense. For example, the target may be an air base which is warned that an attack is about to take place; as the aircraft sequentially leave the base the target value diminishes with the passage of time (later weapons are less threatening than earlier ones). Let m_i denote the number of defensive missiles to be allocated to the ith weapon (in order of arrival) which lands within a distance R of the target, and let B_i denote the probability that this weapon will penetrate: $B_i = (1-\rho)^{m_i}$. Then the expected number of weapons (landing inside a circle of radius R centered on the target) to the first penetrator is

$$E = B_1 + 2B_2(1-B_1) + \dots + jB_j \prod_{i=1}^{j-1}(1-B_i) + (j+1)\prod_{i=1}^{j}(1-B_i) \quad ,$$

if the first j weapons (landing inside the circle) are assigned defensive missiles:

$$\sum_{i=1}^{j} m_i = m \,, \quad m_i \geq 0 \quad \text{for} \quad 1 \leq i \leq j \,, \quad m_{j+1} = 0 \quad .$$

This problem is identical to that of the previous section if one sets $r_i = i$. To maximize E approximately, subject to the constraint that $\prod_{i=1}^{j} B_i = B = (1-\rho)^m$, one may proceed as in the previous section. Substitute $B(B_1B_2 \ldots B_{j-1})^{-1}$ for B_j in the above equation and solve the system of equations $\partial E/\partial m_i = 0$, $i = 1,2,\ldots,j-1$ to obtain the optimum values of $B_1,B_2,\ldots,B_{j-1}$. Select that value of B that makes $B_j = 1$; that is, let $B = B_1B_2 \ldots B_{j-1}$. The recursion formulas of the previous section simplify slightly, since $r_{k+1} - r_k = 1$.

The optimum values of B_i and the expected number of weapons to the first penetrator are tabulated below for those defense resources B such that $B = B_1B_2 \ldots B_{j-1}$, for $1 \leq j \leq 40$.

The second column of this table gives the optimum B_i associated with each of the weapons to which defensive missiles are assigned. (If, for example, the fifteenth offensive weapon is the first one not engaged, then the entries in the second column give $B^*_{14} = 0.5000$, $B^*_{13} = 0.2500,\ldots,B^*_1 = 0.0186$.) The third column gives $-\log_e B$, which is equal to $-m \log_e(1-\rho)$. When ρ is near unity, this is approximately equal to $m\rho$. It would be worthwhile to develop asymptotic expressions for each of these columns.

A numerical illustration may be helpful here. Suppose $\rho = 0.4$ and $m = 7$; then $-\log_e(1-\rho) = .511$. If the third weapon is the first one not engaged, then $B^*_2 = .5000$, and $B^*_1 = .2500$. The total defensive missile stockpile is equal to $m = (2.08)/(.511) = 4.06$, which is allocated $m_2 = .69/.511 = 1.35$ and $m_1 = m - m_2 = 2.71$. If the fourth weapon is the first one not engaged, then $B^*_3 = .5000$, $B^*_2 = .2500$, and $B^*_1 = .1500$. The total defensive missile stockpile is equal to $m = (3.98)/(.511) = 7.80$, which is allocated as follows: $m_3 = .69/.511 = 1.35$, $m_2 = (2.08 - .69)/.511 = 2.71$, and $m_1 = m - m_3 - m_2 = 3.74$. The defensive stockpile is located .786 of the way between 4.06 and 7.80; therefore $m_1 = 2.71 + .786(3.74 - 2.71) = 3.52$, $m_2 = 1.35 + .786(2.71 - 1.35) = 2.42$, and $m_3 = (.786)(1.35) = 1.06$. Of course, since it is impossible to allocate the fractional missiles, a good practical allocation is $m_1 = 4$, $m_2 = 2$ and $m_3 = 1$. Again, round-off can be expected to be a problem.

OPTIMUM ALLOCATION OF DEFENSE RESOURCES AND THE EXPECTED NUMBER OF THE FIRST PENETRATING WEAPON

Number of First Unengaged Offensive Weapon	Probability of Penetration of Engaged Weapons (in Reverse Order)	Total Defensive Missile Stockpile Required (Normalized)	Expected Number of First Penetrator
1	0.5000	0.00	1.00
2	0.2500	0.69	1.50
3	0.1500	2.08	2.13
4	0.1020	3.98	2.81
5	0.0752	6.26	3.52
6	0.0586	8.85	4.26
7	0.0474	11.68	5.01
8	0.0395	14.73	5.77
9	0.0337	17.96	6.54
10	0.0292	21.35	7.32
11	0.0257	24.89	8.11
12	0.0229	28.55	8.90
13	0.0205	32.33	9.69
14	0.0186	36.21	10.50
15	0.0170	40.19	11.30
16	0.0156	44.26	12.11
17	0.0144	48.43	12.92
18	0.0134	52.66	13.73
19	0.0125	56.97	14.55
20	0.0117	61.35	15.37
21	0.0110	65.80	16.19
22	0.0103	70.31	17.01
23	0.0098	74.89	17.83
24	0.0092	79.51	18.66
25	0.0088	84.20	19.48
26	0.0084	88.93	20.31
27	0.0080	93.72	21.14
28	0.0076	98.55	21.98
29	0.0073	103.43	22.81
30	0.0070	108.36	23.64
31	0.0067	113.32	24.48
32	0.0064	118.33	25.31
33	0.0062	123.38	26.15
34	0.0059	128.47	26.99
35	0.0057	133.59	27.83
36	0.0055	138.75	28.67
37	0.0053	143.95	29.51
38	0.0052	149.18	30.35
39	0.0049	154.45	31.20
40	–	159.72	32.05

There exists an even simpler missile allocation strategy which is very nearly optimum. If one has m defensive missiles, allocate approximately m/h of these to each of the first h weapons and none to the (h+1)st weapon. Numerical calculations indicate that the optimum choice of h is about 90 per cent of the first unengaged weapon under the optimum allocation; the loss in the expected number of the first penetrator is only 6 per cent compared with the (continuous) optimum.

EVEN ALLOCATION OF DEFENSE RESOURCES AND THE EXPECTED NUMBER OF THE FIRST PENETRATING WEAPON

Number of First Unengaged Offensive Weapon: Optimum Allocation	Number of First Unengaged Offensive Weapon: Even Allocation	Probability of Penetration for Any Engaged Weapon (in Reverse Order)	Expected Number of First Penetrator
2	2	0.5000	1.50
3	3	0.3535	2.06
4	4	0.2658	2.67
5	4	0.1342	3.31
6	5	0.1095	4.02
7	6	0.0966	4.72
8	7	0.0858	5.43
9	8	0.0769	6.15
10	9	0.0695	6.87
12	11	0.0574	8.32
14	12	0.0365	9.83
16	14	0.0332	11.35
18	16	0.0299	12.87
20	18	0.0271	14.40
25	22	0.0181	18.27
30	27	0.0155	22.20
40	36	0.0104	30.14

3.2.2 An Exact Procedure for Maximizing the Expected Rank

If the defensive missile reliability ρ is close to 1, it is desirable to replace the missile allocation procedure presented above with one which makes integer allocations. The general form of a typical allocation is clear, provided m is not too large: for the first j offensive weapons, one should allocate two defensive missiles apiece, and for the last i offensive weapons, one should allocate one defensive missile apiece.

It is possible to determine the optimal allocation by dynamic programming; however, the problem may also be treated directly. Set

$$\begin{aligned} q &= 1 - p \ , \\ x &= 1 - p(1-\rho) \ , \\ y &= 1 - p(1-\rho)^2 \ , \\ z &= 1 - p(1-\rho)^3 \ . \end{aligned}$$

Thus q, x, y, and z are the probabilities a given offensive weapon penetrates if 0, 1, 2, or 3 defensive missiles respectively are assigned to it.

Suppose the defense allocates one missile to each of the first $m+1$ weapons. Then the expected rank of the first penetrator is

$$\begin{aligned} E(m+1) &= 1 + x + \ldots + x^{m+1} + qx^{m+1} + q^2x^{m+1} + \ldots \\ &= \frac{1 - x^{m+1}}{1 - x} + \frac{x^{m+1}}{1 - q} \\ &= \left(1-\rho x^{m+1}\right)/p(1-\rho) \ . \end{aligned}$$

Suppose the defense instead allocates two missiles to the first weapon and one missile to each of the next $m-1$ weapons. Then the expected rank of the first penetrator is

$$\begin{aligned} E(1,m-1) &= 1 + y + yx + yx^2 + \ldots + yx^{m-1} + yx^{m-1}q + yx^{m-1}q^2 + \ldots \\ &= 1 + y\,\frac{1 - x^{m-1}}{1 - x} + \frac{yx^{m-1}}{1 - q} \\ &= 1 + \left(y-yx^{m-1}\rho\right)/p(1-\rho) \ . \end{aligned}$$

The second course of action will be preferable for the defense whenever $E(1,m-1) > E(m+1)$. After some calculation one finds that

$$E(1,m-1) - E(m+1) = \rho\left(1 - x^{m-1}(1+\rho-p+p\rho)\right) \ .$$

Therefore $E(1,m-1) > E(m+1)$ whenever

$$x^{m-1} < 1/(1+\rho-p+p\rho) \ ,$$

so that

$$m > -\log(1+\rho-p+p\rho)/\log x + 1 \ ,$$

or equivalently,

$$m \geq \lfloor -\log(1+\rho-p+p\rho)/\log x \rfloor + 2 \quad .$$

In what follows, the last expression will be designated by m_0.

Let M be the defensive stockpile. Assume either that M is not too large or that for some reason the defense cannot assign more than two missiles to any weapon. Then the complete solution is as follows. If $M \leq m_0$, assign one missile each to the first M weapons. If $M > m_0$, assign two missiles each to the first $\lfloor (M-m_0+1)/2 \rfloor$ weapons and one missile each to the next $M - \lfloor (M-m_0+1)/2 \rfloor$ weapons. A significant special case is that in which $p = 1$, when one has simply

$$m_0 = \lfloor \log(1/2)/\log \rho \rfloor + 1 \quad .$$

For example, suppose $\rho = 0.8$, $p = 1$. Then $m_0 = \lfloor 3.11 \rfloor + 1 = 4$. Thus as the stockpile grows, the successive allocations are as follows:

$$(0,1),(0,2),(0,3),(0,4),(1,3),(1,4),(2,3),(2,4),\ldots$$

Now consider the possibility that three missiles may be assigned to an offensive weapon. Suppose the defense assigns two missiles each to the first n weapons and one each to the next m weapons. Then the expected rank of the first penetrator is

$$\begin{aligned} E(n,m) &= 1 + y + y^2 + \ldots + y^{n-1} + y^n + xy^n + x^2y^n + \ldots + x^{m-1}y^n \\ &\qquad + x^m y^n + qx^m y^n + q^2 x^m y^n + \ldots \\ &= \frac{1-y^n}{1-y} + y^n \frac{1-x^m}{1-x} + \frac{x^m y^n}{1-q} \\ &= \left(\frac{1}{1-\rho} - \frac{y^n \rho}{1-\rho} - x^m y^n \rho\right) p(1-\rho) \quad . \end{aligned}$$

If the defense assigns three missiles to the first weapon, two each to the next n, and one each to the next m, the expected rank of the first penetrator is

$$\begin{aligned} E(1,n,m) &= 1 + z + zy + zy^2 + \ldots + zy^{n-1} + zy^n + zy^n x + zy^n x^2 + \ldots \\ &\qquad + zy^n x^{m-1} + zy^n x^m + zy^n x^m q + zy^n x^m q^2 + \ldots \end{aligned}$$

$$= 1 + z\frac{1 - y^n}{1 - y} + zy^n\frac{1 - x^m}{1 - x} + \frac{zy^nx^m}{1 - q}$$

$$= 1 + \left(\frac{z}{1 - \rho} - \frac{zy^n\rho}{1 - \rho} - zy^nx^m\rho\right)\Big/p(1-\rho) \quad .$$

From these formulas it is possible to compare $E(1,n,m_0-1)$ with $E(n+1,m_0)$ and $E(1,n-1,m_0)$ with $E(n+1,m_0-1)$. These are the most crucial comparisons, although the possibility that other comparisons may sometimes be necessary cannot be ruled out.

First, consider the comparison of $E(1,n,m_0-1)$ with $E(n+1,m_0)$. Using the above formulas one has, after some calculation,

$$E(1,n,m_0-1) - E(n+1,m_0) = \rho\left(1 - y^n\rho - y^nx^{m_0-1}\left(1+\rho-\rho^2-p(1-\rho)^2\right)\right) \quad .$$

Therefore $E(1,n,m_0-1) > E(n+1,m_0)$ whenever

$$y^n < 1\Big/\left(\rho + x^{m_0-1}\left(1+\rho-\rho^2-p(1-\rho)^2\right)\right) \quad ,$$

so that

$$n \geq \left[-\log\left(\rho + x^{m_0-1}\left(1+\rho-\rho^2-p(1-\rho)^2\right)\right)\Big/\log y\right] + 1 \quad .$$

This expression will be denoted by n_0. If $p = 1$,

$$n_0 = \left[-\log\left(\rho + \rho^{m_0}(3-2\rho)\right)\Big/\log\left(2\rho-\rho^2\right)\right] + 1 \quad .$$

Second, compare $E(1,n-1,m_0)$ with $E(n+1,m_0-1)$. One has

$$E(1,n-1,m_0) - E(n+1,m_0-1)$$

$$= \rho\left(1 - y^{n-1}\left(1+\rho-p(1-\rho)^2\right) + y^{n-1}x^{m_0-1}\rho^3\right) \quad .$$

Therefore $E(1,n-1,m_0) > E(n+1,m_0-1)$ whenever

$$y^{n-1} < 1\Big/\left(1 + \rho - \rho(1-\rho)^2 - x^{m_0-1}\rho^2\right) ,$$

so that

$$n \geq \left[-\log\left(1 + \rho - p(1-\rho)^2 - x^{m_0-1}\rho^2\right) \ \log y\right] + 2 .$$

This expression will be denoted by n_1. If $p = 1$,

$$n_1 = \left[-\log\left(\rho\left(3-\rho-\rho^{m_0}\right)\right)\Big/\left(\log 2\rho-\rho^2\right)\right] + 2 .$$

Suppose that M is not too large or that the defense cannot assign more than three missiles. Then if $-1 \leq n_1 - n_0 \leq 2$, the above results suffice to determine the optimal allocation. The solution is hard to state concisely, but can be given as follows. Form $M - (n_0 + n_1 + m_0)$. If this expression is negative, no weapon is assigned three missiles and the solution is as before. Otherwise, define k and r by

$$M - \left(n_0 + n_1 + m_0\right) = 3k + r , \quad 0 \leq r \leq 2 .$$

Set $i = n_1 - n_0$. Then the allocation is as follows. The number of weapons assigned three, two and one missiles are $k + a_{ir}$, $n_0 + b_{ir}$, $m_0 + c_{ir}$ respectively, where a_{ir}, b_{ir}, c_{ir} are given by the following table:

$i = -1$:

r	a_{ir}	b_{ir}	c_{ir}
0	1	-2	0
1	0	0	0
2	1	-1	0

$i = 0$:

r	a_{ir}	b_{ir}	c_{ir}
0	0	0	0
1	1	-1	0
2	1	0	-1

$i = 1$:

r	a_{ir}	b_{ir}	c_{ir}
0	0	1	-1
1	1	0	-1
2	1	0	0

$i = 2$:

r	a_{ir}	b_{ir}	c_{ir}
0	1	0	-1
1	0	2	-1
2	1	1	-1

This solution is complete, provided $-1 \leq n_1 - n_0 \leq 2$. Situations in which $n_1 - n_0$ exceeds this range appear rather uncommon. If that range is exceeded, it is necessary to compare $E(2,n,m)$ with $E(n+3,m)$, where $m = m_0$ or $m_0 - 1$; even more remote comparisons might be necessary on rare occasions. Thus, although the above analysis should be satisfactory in most cases, a more complete analysis of the problem would be very desirable. Another shortcoming of the current knowledge of the problem is that no solution has been given which permits more than three missiles to be assigned to an offensive weapon. There seems some hope that an overall solution might be given; also, since $E(k_1,\ldots,k_r)$ is fairly easily calculated in any specific case, a solution to any specific problem can be found by trial and error. In addition, approximations to the solution exist; however, at present, if it is necessary to find an exact solution in which the allocations to individual weapons are expected to be large, it is probably best to use dynamic programming.

The following table summarizes the defense strategies for $p = 1$ and selected values of ρ, assuming no weapon is assigned more than three missiles. For each ρ, the table gives m_0, n_0, n_1, i and $\overline{M}$, the smallest defense stockpile for which the first weapon receives three weapons. (Note, however, that if the stockpile is $\overline{M} + 1$, the first weapon may not receive three missiles; note also that sometimes $\overline{M} = m_0 + n_0 + n_1 + 1$.)

ρ	m_0	n_0	n_1	i	$\overline{M}$
0.7	2	5	4	-1	11
0.75	3	6	6	0	16
0.8	4	8	10	2	22
0.85	5	16	18	2	39
0.9	7	39	39	0	86
0.95	14	159	159	0	333

To illustrate the use of this table, consider again the case $\rho = .8$, $p = 1$, so $i = 2$. If $M = 22$, $k = 0$, $r = 0$, and the allocation is $(0+1, 8+0, 4-1) = (1,8,3)$. If $M = 23$, $k = 0$, $r = 1$, and the allocation is $(0,10,3)$. Likewise if $M = 24$, the allocation is $(1,9,3)$. After this, the last two numbers of the allocation are periodic with period three. Thus as the stockpile grows, the successive allocations are as follows:

$$(0,0,1),(0,0,2),(0,0,3),(0,0,4),(0,1,3),(0,1,4),\ldots,(0,8,4),(0,9,3),$$
$$(1,8,3),(0,10,3),(1,9,3),(2,8,3),(1,10,3),(2,9,3),(3,8,3),\ldots$$

3.2.3 A Constant Value Decrement Criterion

Everett (1968) suggests an alternative defense strategy to be used when the attack is sequential and of unknown size. Instead of maximizing the expected number of weapons to the first penetrator, he designs a defense strategy which makes the probability of target destruction proportional to the attack size, up to the point of defensive missile exhaustion. It is immaterial to the offense what attack level is selected: the increase in target destruction probability achieved by assigning one additional weapon to the target is always the same. This strategy is sometimes called a constant value decrement (or CVD) doctrine. Of course, as in any continuous model, there is a round-off problem.

Everett assumes a somewhat more general model than that of Section 3.2.1; the probability of target kill by an unintercepted weapon is equal to a constant p. Both the probability of target kill p and the defensive missile reliability ρ are known by the defense. Let k be the increase in the probability of target destruction when one more weapon is directed against it; the CVD doctrine must assure the defense that the probability of target kill by i weapons is ik for $i \leq m$ and near unity for n, where n is the number of weapons needed to exhaust the defensive missile stockpile, m. Let P_i denote the probability of target kill by an attack of i weapons, and let m_j be the number of defensive missiles assigned to the jth weapon to arrive, $1 \leq j \leq i$.

$$P_i = 1 - \prod_{j=1}^{i} \left(1-p(1-\rho)^{m_j}\right) = ik \quad .$$

Solving for ρ, one finds that

$$(1-\rho)^{m_i} = \frac{k}{(1-ik+k)p} \quad .$$

Since the probability of target kill increases in steps of k until reaching the vicinity of unity at $i = n$, one sets $k = 1/n$ and substitutes this into the above equation. Solving for m_i, one obtains

$$m_i = -\log\left((1-i/n)p\right)/\log(1-\rho) \ , \qquad i = 1,2,\ldots,n$$

$$m = \sum_{i=1}^{n} m_i \ ,$$

which gives nearly the optimum defensive missile allocation. If $p < 1$, one finds that $m_n < 0$; in practice, m_n should be close enough to zero not to cause a problem.

A numerical illustration may be helpful here. In order to compare this allocation with the earlier one, assume that $p = 1$, $\rho = 0.4$ and $m = 7$. If n is assumed to be 4, then $m_1 = 2.71$, $m_2 = 2.14$, $m_3 = 1.35$ and $m_4 = 0$; the total defensive stockpile needed is 6.20. If n is assumed to be 5, then $m_1 = 3.14$, $m_2 = 2.71$, $m_3 = 2.14$, $m_4 = 1.35$ and $m_5 = 0$; the total defensive stockpile needed is 9.34. Interpolating linearly between these solutions, one finds that the optimum allocation corresponding to a stockpile of seven defensive missiles is $m_1 = 2.82$, $m_2 = 2.28$, $m_3 = 1.55$ and $m_4 = 0.35$. Interpolating similarly between 1/5 and 1/4, the value of k is found to be 0.238.

This allocation is quite different from the one (in Section 3.2.1) which maximizes the expected number of weapons to the first penetrator. However, the reduction in the expected number of weapons to the first penetrator is quite small:

$$E = k + 2k + \ldots + nk = kn(n+1)/2 = (n+1)/2 \ .$$

For n equal to 5, $E = 3$; for n equal to 4, $E = 2.5$. Interpolating, E is equal to 2.624, very little less than the theoretical maximum of $2.13 + 0.768\ (2.81-2.13) = 2.68$ obtained using the allocation given in Section 3.2.1.

Obviously, it is impossible to allocate fractional defensive missiles; therefore, any reliable strategy will depart somewhat from the ideal one in which the expected damage is proportional to the number of offensive weapons. To get around this difficulty, Everett suggests that the defense allocate $[m_i]$ missiles with probability p_i and $[m_i] + 1$ missiles with probability $1 - p_i$, where p_i is chosen so that

$$p_i(1-\rho)^{[m_i]} + (1-p_i)(1-\rho)^{[m_i]+1} = (1-\rho)^{m_i} \quad .$$

Even though any specific realization will not be a CVD doctrine, the average over all possible realizations will be. Solving for p_i, one finds that

$$p_i = (1/\rho)\left((1-\rho)^{m_i-[m_i]}-(1-\rho)\right) \quad .$$

Note that this strategy does not use a fixed number of defensive missiles; in general, extra missiles will be required to take care of the variability inherent in Everett's randomized allocation. (For example, in the example given above, nine missiles instead of seven must be provided at the target.) A randomized allocation will typically result in unused defensive missiles; hence, it is inferior (in the sense of maximizing the expected number of weapons to the first penetrator) to many nonrandomized strategies which fully use the required extra missiles. This objection is mitigated if there is more than one target being defended and one has the option of shifting defensive missiles from one target to another (as needed by the randomization).

However, a more serious objection to randomization is that it will generally lead to firing doctrines in which the number of missiles directed at the ith incoming weapon is greater than the number of missiles directed at the (i+1)st incoming weapon. Clearly, a firing doctrine such as this is inferior (in the sense described above) to one which reverses these allocations, even if the latter is less faithful to the CVD doctrine.

Everett has tabulated the parameter k, the increase in probability of target destruction per offensive weapon, for a number of different values of m, ρ and p when a constant value decrement defense strategy is used. This table is given on the next page

3.2.4 Known Distribution on Attack Size

Consider now a somewhat different allocation problem. Assume, in particular, that the defense knows the probability distribution of the attack size: the probability that the offense will attack with i or more weapons inside the lethal radius, R, is equal to p_i, $i = 1,2,\ldots,n$, and $p_{n+1} = 0$. It is assumed that offensive weapons arrive one at a time; when the ith weapon arrives, m_i missiles are assigned to it from a stockpile of m missiles. Instead of maximizing the expected number of weapons to the first penetrator, one allocates missiles to minimize the expected number of penetrators: that is, one wishes to select $(m_1,m_2,\ldots,m_n)$, where $\Sigma m_i = m$, so as to minimize

INCREASE IN PROBABILITY OF TARGET DESTRUCTION
PER OFFENSIVE WEAPON USED

(For Weapons Engaged by the Defense)

Defense Stockpile m										
	p	1.0	1.0	1.0	1.0	1.0	0.5	0.2	0.1	0.05
	ρ	0.5	0.6	0.7	0.8	0.9	0.8	0.8	0.8	0.8
20		.107	.091	.076	.065	.053	.051	.034	.026	.018
50		.058	.048	.040	.033	.027	.028	.020	.015	.011
100		.036	.029	.024	.020	.016	.016	.013	.010	.008
200		.021	.017	.014	.012	.008	.010	.008	.006	.005
500		.010	.008	.007	.006	.004	.005	.004	.003	.003
1000		.006	.005	.004	.003	.002	.003	.002	.002	.002

$$E = \sum_{i=1}^{n} p_i (1-\rho)^{m_i} ,$$

where ρ is the defensive missile reliability. The reduction in the expected number of penetrators resulting from adding the jth missile to the ith offensive weapon is $p_i\left\{(1-\rho)^{j-1} - (1-\rho)^j\right\} = R(i,j)$. Note that $R(i,j) \geq R(i,j+1)$; in other words, each new missile contributes less to the defense than the preceding one. To obtain the optimum allocation, one simply assignes defensive missiles one at a time where they will do the most good (that is, achieve the greatest reduction in the expected number of penetrators). For example, assume $p_1 = 1$, $p_2 = 2/3$, $p_3 = 0$ and $\rho = 1/2$. The first missile is assigned to the first weapon since $R(1,1) = 1/2$ and $R(2,1) = 1/3$; the second missile is assigned to the second weapon since $R(1,2) = 1/4$ and $R(2,1) = 1/3$; the third missile is assigned to the first weapon since $R(1,2) = 1/4$ and $R(2,2) = 1/6$; and so on, until the stockpile is exhausted. The fact that this necessarily leads to the optimal allocation follows from the fact that $R(i,j) \geq R(i,j+1)$.

There is another method which can be useful when it is desired to find the allocation of missiles corresponding to a single stockpile m, rather than a range of stockpiles. One can always solve the equations

$$p_i - p_{i+1} = (1-\rho)^{x_i + y_i} , \quad i = 1,2,\ldots,n$$

for x_i and y_i, where x_i is a nonnegative integer and $0 \le y_i \le 1$. Let A denote the greatest integer contained in $(\Sigma x_i + m)/n$, and B be equal to $\Sigma x_i + m - nA$. Then a tentative optimum allocation of defensive missiles is

$$m_i = A - x_i \qquad \text{for} \quad 1 \le i \le n - B \quad ,$$

$$m_i = A + 1 - x_i \qquad \text{for} \quad n - B + 1 \le i \le n \quad .$$

It will generally happen that $m_i < 0$ for $i > n'$; this implies that the corresponding probability $p_i - p_{i+1}$ of i attacking weapons is too small to allocate a defensive missile. One should then repeat this procedure with n' in place of n, and so on until a nonnegative set of allocations is obtained. The repetitive nature of this procedure diminishes its usefulness for machine computation; however it can be arranged in a form that is rather suitable for hand computation. A closely related approach can be found in Dym and Schwartz (1969).

If one is interested in defensive missile allocations corresponding to all stockpiles less than or equal to m, there are various ways of expressing these allocations in a table, given a set of values $p_1, p_2, \ldots, p_n$. Although not the most compact, the following form of table is useful, which enables one to read off, one at a time, the successive allocations to offensive weapons: each row corresponds to the number of offensive weapons previously assigned missiles $(0,1,\ldots,n-1)$, each column corresponds to the number of defensive missiles left in the stockpile $(0,1,2,\ldots,m)$, and each entry tells how many missiles should be allocated to the current offensive weapon. The allocation corresponding to a given stockpile size m can be easily read off from the table. Suppose that the allocation to the first weapon (in row 0, column m) is m_1; then one goes to row 1, column $m - m_1$ for the allocation to the second weapon. If this is m_2, then one goes to row 2, column $m - m_1 - m_2$ for the allocation to the third weapon, and so on.

One way of constructing this table is to use dynamic programming, while turning the problem around by building up the allocation table from the last engaged weapon to the first. Let $m(j,i)$ denote the number of defensive missiles to be assigned to the $(j+1)$st attacker when i defensive missiles remain; let $g(j,i)$ denote the expected number of future penetrators if j attackers have already been observed, i missiles remain, and the allocation table is used to assign missiles to weapons arriving later. Clearly, $m(j,0) = 0$ and $m(j,1) = 1$ for $0 \le j \le n - 1$; the corresponding $g(j,i)$ are

$$g(j,0) = \sum_{k=j+1}^{n} p_k/p_j \quad ,$$

$$g(j,1) = \left((1-\rho)+g(j+1,0)\right)p_{j+1}/p_j \quad .$$

The g(j,i) and m(j,i) are recursively calculated using the equations

$$g(j,i) = \min_{1 \le k \le i} \left((1-\rho)^k + g(j+1,i-k)\right)p_{j+1}/p_j \quad ,$$

$$m(j,i) = \text{minimizing value of } k \quad .$$

It is not too diffi ult to generalize these recursive equations to include replacement of early-launch failures of defensive missiles, or to allow the defensive missile reliability to vary with i and j.

3.2.5 The Selection of an Attack Distribution

In order to derive these missile allocations, one must know the probability p_i that the offense will attack the target with i or more weapons. At least two probability density functions merit consideration:

1. If the offense has a weapon launch probability of s, and attempts to launch n weapons, then the number on target is given by the binomial probability density function:

   $$p_i - p_{i+1} = \binom{n}{i} s^i (1-s)^{n-i} \quad .$$

   The defense must estimate both s and n beforehand, not necessarily an easy task. On the other hand, for many reasons, it is extremely desirable that the defense make such an estimate. It is possible to express the uncertainty about s and n by replacing the above density by some linear combination of binomial densities.

2. Perhaps the offense will fire weapons at a target until it believes that it has launched a "successful" weapon (here "successful" may be defined as one predicted to land very near the target, or one which the offense believes has penetrated the defense). If the probability of calling a weapon "successful" is s, and is independent from weapon to weapon, then the number of "unsuccessful" weapons allocated to the target is given by a geometric probability density function:

   $$p_i - p_{i+1} = (1-s)^i s \quad .$$

Again, the defense must estimate s beforehand. The geometric distribution has one property which may strongly appeal to the enemy, quite apart from any success-prediction scheme — the conditional probability density function of i more weapon arrivals, given that k have already arrived, is independent of k. In other words, the defense can gain no information about the probable future attack size during the course of the attack.

If the geometric attack distribution is assumed, the allocation table described earlier takes a particularly simple form — all rows are identical. In order to obtain some idea of the sensitivity of the defensive allocation to the assumption of the offensive parameter s, a sample allocation table has been prepared (see next page). In this table, the independent variable has been transformed from s to $E = (1-s)/s$.

3.3 DEFENSE STRATEGIES AGAINST A SEQUENTIAL ATTACK BY WEAPONS OF UNKNOWN LETHAL RADIUS

Consider now an entirely new class of offensive weapon attacks. Suppose that the defense knows the size, n, of the attack, but not the lethal radius, R, of the offensive weapons, which arrive one at a time. To simplify the problem, assume that defensive missiles have perfect reliability ($\rho = 1$), so that only one defensive missile need be assigned to an offensive weapon. The defense's problem can be summarized as follows. An offensive weapon appears, and the defense observes that it will impact at a distance r from the target if it is not intercepted. The defense has m missiles available, and knows exactly n additional offensive missiles will arrive later. Should he destroy this weapon, or should he save his missiles for potentially more threatening (smaller values of r) later weapons?

The defense strategy is influenced by his knowledge of the probability density function of impact-points of the offensive weapons. To simplify the analysis, consider two extreme cases:

a. The defense knows nothing about the density function, and must learn from observations of actual r_i early in the attack which ones are "close" and which ones "distant."

b. The defense knows the exact density function, so that he can calculate for any observed r the probability that a random offensive weapon will land closer to the target.

Obviously, the defense should perform better in the second case than in the first.

ALLOCATION OF DEFENSIVE MISSILES AGAINST A GEOMETRIC ATTACK

Number of Missiles Left	Defensive Missile Reliability 0.5						Defensive Missile Reliability 0.8					
	E = 1		E = 2		E = 4		E = 1		E = 2		E = 4	
	m	g	m	g	m	g	m	g	m	g	m	g
0	0	1.00	0	2.00	0	4.00	0	1.00	0	2.00	0	4.00
1	1	.75	1	1.67	1	3.60	1	.60	1	1.47	1	3.36
2	1*	.63	1	1.44	1	3.28	1	.40	1	1.11	1	2.85
3	2	.50	2	1.28	1	3.02	1	.30	1	.87	1	2.44
4	2*	.44	2	1.13	1	2.82	2	.22	1	.72	1	2.10
5	2*	.38	2	1.02	2	2.62	2	.17	2	.61	1	1.85
6	3	.31	2	.92	2	2.45	2	.13	2	.50	1	1.64
7	3*	.28	3	.84	2	2.29	2	.11	2	.42	1	1.47
8	3*	.25	3	.76	2	2.16	2	.09	2	.36	1	1.34
9	3*	.22	3	.68	2	2.03	3	.07	2	.31	2	1.22

*An alternative allocation of one more missile is also optimum.

In order to determine an optimum strategy, one must have a criterion of effectiveness. Again, the probability of target destruction cannot be used directly, so substitutes must be sought. Two reasonable possibilities are:

a. Maximize the probability that the offensive weapon landing nearest the target is assigned a defensive missile.

b. Maximize the expected total score of the offensive weapons destroyed, where the score of the ith weapon is the probability that a random offensive weapon will land farther from the target than the ith weapon did.

Note that the first criterion does not guarantee that all defensive missiles will be used, since the defense need not allocate a defensive missile to any offensive weapon unless it is the closest one to the target thus far observed. On the other hand, the second criterion insures that all defensive missiles will be used.

It should be noted that the models described in the next two sections were derived for applications unrelated to missile allocation and target defense.

3.3.1 Maximizing the Probability of Intercepting the Nearest Weapon

In an asymptotic sense, the best defensive strategy under the first criterion has been derived by Gilbert and Mosteller (1966), for unknown probability density functions of impact-points. If the probability density function of impact-points is unknown, Gilbert and Mosteller show that the optimum strategy must have the following asymptotic form (for n large):

> Assume that there are m defensive missiles available, and let the sequentially-observed offensive weapon miss-distances be denoted by $r_1, r_2, \ldots, r_n$. The ith weapon is assigned a defensive missile if its miss-distance r_i satisfies any of the following m criteria, based on the m constants $\alpha_1 < \alpha_2 < \ldots < \alpha_m$: Set $k = [\alpha_j n]$ and $r = \min(r_1, r_2, \ldots, r_k)$. Then r_i satisfies the jth criterion if $r_i < r$, but $r_{k+1}, r_{k+2}, \ldots, r_{i-1}$ are all $\geq r$.

In other words, the defense plays m simultaneous games of the following form: observe the smallest miss-distance in a fraction α_j of the attack, and then assign a defensive missile to the first offensive missile appearing with a smaller miss-distance. This enables the defense to allocate a maximum of m missiles, but it is possible that not a single missile will be fired using this strategy (this will happen if the smallest r_i in the attack occurs in the initial fraction of the attack α_1). [1]

The optimum fractions α_j are quite tedious to compute; Gilbert and Mosteller tabulate them for $1 \leq m \leq 8$. For m = 8, the eight fractions are

$\alpha_1 = .0172$	$\alpha_5 = .0910$
$\alpha_2 = .0259$	$\alpha_6 = .1411$
$\alpha_3 = .0391$	$\alpha_7 = .2231$
$\alpha_4 = .0594$	$\alpha_8 = .3679$

For any $m < 8$, the α_i are to be taken from the last m entries of this table. For example, for m = 1, the fraction α_1 is equal to $1/e = .3679$; for m = 2, the two fractions are $\alpha_1 = 1/e^{3/2} = .2231$ and $\alpha_2 = 1/e = .3679$. It is much easier for the defense to use a nonoptimum strategy of the following asymptotic form (for n large):

> Assume that there are m defensive missiles available, and let the sequentially-observed offensive weapon miss-distances be noted by $r_1, r_2, \ldots, r_n$. The ith weapon is assigned a defensive missile if $r_i < \min(r_1, r_2, \ldots, r_{i-1})$, if $i > \alpha n$, and if there are defensive missiles remaining.

In other words, the defense observes the smallest miss-distance in a fraction α of the attack, and then assigns defensive missiles to the first m offensive weapons which set new record minimum miss-distances thereafter. Gilbert and Mosteller show that the optimum value of α is $\exp\left(-(m!)^{1/m}\right)$, and the corresponding probability of assigning a defensive missile to the weapon with the smallest miss-distance is

$$P = \exp\left(-(m!)^{1/m}\right) \sum_{i=1}^{m} (r!)^{i/m}/i! \quad .$$

The table on the next page compares the probabilities of attacking the closest offensive weapon for these two strategies.

Now suppose that the probability density function of impact-points, p(r), is known. Gilbert and Mosteller do not derive the optimum strategy, but instead suggest a nonoptimum strategy of the following asymptotic form (for n large), which depends on a single parameter k/n:

> Assume that there are m defensive missiles available, and let the sequentially-observed offensive weapon miss-distances be denoted by $r_1, r_2, \ldots, r_n$. The ith

m	α	Probability of Attacking Closest Offensive Weapon: Optimum	Non-Optimum
1	0.368	0.368	0.368
2	0.243	0.591	0.587
3	0.162	0.732	0.726
4	0.109	0.823	0.817
5	0.074	0.883	0.877
6	0.050	0.922	0.917
7	0.034	0.948	0.944
8	0.023	0.965	0.962

weapon is assigned a defensive missile if $r_i \leq r^*$,

where r^* is defined by the equation $\frac{k}{n} = \int_{-\infty}^{r^*} p(r)dr$.

After this inequality has been satisfied for m missiles, the stockpile is exhausted.

In other words, the defense assigns missiles to the first m offensive weapons having miss-distances less than a critical value. Gilbert and Mosteller have determined the optimum choice of k and the associated probability of assigning a defensive missile to the weapon with smallest miss-distance, for $1 \leq m \leq 10$.

m	k	Probability of Attacking Closest Offensive Weapon
1	1.503	0.5174
2	2.435	0.7979
3	3.485	0.9254
4	4.641	0.9753
5	5.890	0.9926
6	7.225	0.9980
7	8.637	0.99949
8	10.121	0.99988
9	11.672	0.99997
10	13.284	0.99999

This strategy would be improved by replacing it with an optimum strategy analogous to the one used when the probability density function of weapon impact-points is unknown: assign defensive missiles if the miss-distance r_i is less than a critical value and also is less than all earlier miss-distances $r_1, r_2, \ldots, r_{i-1}$.

How good is the simple strategy depending only on k/n? Gilbert and Mosteller derive the optimum strategy for $m = 1$, finding that the probability of attacking the closest offensive weapon approaches 0.5802 as n approaches infinity. This should be compared with the probability of 0.5174 in the above table. As a rule of thumb, for small values of m, a knowledge of the probability density function of missile impact-points is worth about $m/2$ defensive missiles, providing corresponding optimum or non-optimum defense strategies are used.

3.3.2 Maximizing the Total Score of the Intercepted Weapons

Consider now the best defensive strategy under the somewhat more realistic second criterion: maximize the expected total score of the offensive weapons destroyed, where the score of a given weapon is the probability that an offensive weapon drawn at random from the impact-point population will land farther from the target than the given weapon did. This strategy was derived by Gilbert and Mosteller (1966).

Assume, as usual, that the defensive missile reliability, ρ, is equal to unity. Assume that the defense knows $p(r)$, the probability density function of impact-points. The optimum defense strategy has the following form:

> Suppose that there are $t \leq m$ defensive missiles remaining in the stockpile, and $k \leq n$ offensive weapons yet to appear in the attack. When the first of the k offensive weapons appears, with miss-distance r, assign a defensive missile if $r \leq r(k,t)$, where $r(k,t)$ is defined by $U(k,t) = \int_{-\infty}^{r(k,t)} p(r)dr.$

Let $tE(k,t)$ denote the expected score of the offensive weapons destroyed by the final t defensive weapons; in other words, $E(k,t)$ is the average value of the t probabilities that a random offensive weapon exceeds the t observed offensive weapon miss-distances. Then one can write down the following iterative equation:

$$tE(k,t) = \big(1-U(k,t)\big)\Big(\big(1+U(k,t)\big)/2+(t-1)E(k-1,t-1)\Big) + U(k,t)tE(k-1,t) \quad .$$

To determine the maximum value of $E(k,t)$, differentiate the above expression with respect to $U(k,t)$, set this equal to zero, and solve for $E(k,t)$. This yields

$$U^*(k,t) + (t-1)E^*(k-1,t-1) = tE^*(k-1,t) \quad .$$

One can easily derive the optimum $E^*(k,t)$ and $U^*(k,t)$ recursively with the aid of the initial conditions $U^*(k,k) = 0$, $1 \le k \le n$, $E^*(k,k) = 1/2$, $1 \le k \le n$, and the iterative equation above. When $m = 1$, this specializes to $U^*(k,1) = E^*(k-1,1)$, and the iterative equation becomes $E^*(k,1) = \left(1+E^*(k-1,1)^2\right)/2$.

It appears difficult to obtain simple analytic expressions for $E^*(k,t)$ and $U^*(k,t)$. Gilbert and Mosteller approximate $E^*(k,1)$ by the expression

$$E^*(k,1) = 1 - 2/\left(k+\log_e(k+1)+1.767\right)$$

for large values of k. As k and t approach infinity in such a way that t/k approaches f, it is conjectured that $U^*(k,t)$ approaches $1-f$. One can prove that $U^*(a+b,a) = 1 - U^*(a+b,b)$, which cuts down somewhat on the tabulation of $U^*(k,t)$.

The tables on the next two pages give selected values of $U^*(k,t)$ and $E^*(k,t)$. In order to save space, not all integral values of k and t have been included. The reader who needs intermediate values can easily perform two-way interpolation. A less extensive table of $tE^*(k,t)$ is given in Gilbert and Mosteller (1966), for $t = 1(1)3$ and $k = 1(1)10(10)50(50)100(100)500(500)1000$.

How well does the above strategy perform relative to what could be accomplished with perfect information? If one could examine all k miss-distances of the remaining offensive weapons, then one could attack those offensive weapons with the t smallest miss-distances. The expected score of the t smallest miss-distances is

$$tE_0(k,t) = \sum_{i=1}^{t} \left(1-i/(k+1)\right) = t - t(t+1)/2(k+1) \quad .$$

For comparative purposes, the quantity $E_0(50,t)$ has been included at the bottom of the table of $E^*(k,t)$.

3.4 DEFENSE STRATEGIES AGAINST A SEQUENTIAL ATTACK CONTAINING EXACTLY ONE WEAPON MIXED WITH DECOYS

Suppose the defense knows that the attack consists of exactly one offensive weapon and (n-1) offensive decoys. The offensive weapon is characterized by a single observation (a real number) drawn from a population with a probability density function $f_w(x)$ known to the defense, and the decoys are characterized by (n-1) observations (real numbers) drawn from a probability density

DECISION PROBABILITIES $U^*(k,t)$

k	Remaining Stockpile Size t 1	2	3	4	5	6	7	8	9	10	12	14	16	18	20	22
1																
2	.500															
3	.625															
4	.695	.500														
5	.742	.579														
6	.775	.634	.500													
7	.800	.676	.558													
8	.820	.708	.603	.500												
9	.836	.735	.639	.546												
10	.850	.757	.669	.584	.500											
11	.861	.775	.694	.616	.539											
12	.871	.791	.716	.643	.571	.500										
13	.879	.804	.734	.666	.599	.533										
14	.886	.816	.750	.687	.624	.562	.500									
15	.893	.827	.765	.705	.646	.587	.529									
16	.899	.836	.777	.720	.665	.610	.555	.500								
17	.904	.844	.789	.735	.682	.630	.578	.526								
18	.908	.852	.799	.748	.698	.648	.598	.549	.500							
19	.913	.858	.808	.759	.711	.664	.617	.570	.523							
20	.916	.865	.816	.770	.724	.679	.634	.589	.544	.500						
22	.923	.876	.831	.788	.746	.705	.663	.622	.581	.540						
24	.929	.885	.844	.804	.765	.727	.688	.651	.613	.575	.500					
26	.934	.893	.854	.817	.781	.745	.710	.675	.640	.605	.535					
28	.938	.900	.864	.829	.795	.762	.729	.696	.663	.630	.565	.500				
30	.942	.906	.872	.840	.808	.776	.745	.714	.683	.653	.591	.530				
32	.945	.911	.879	.849	.819	.789	.760	.730	.701	.672	.615	.557	.500			
34	.948	.916	.886	.857	.828	.800	.772	745	.717	.690	.635	.581	.527			
36	.951	.920	.892	.864	.837	.810	.784	58	.732	.706	.654	.603	.551	.500		
38	.953	.924	.897	.871	.845	.820	.794	.70	.745	.720	.671	.622	.573	.524		
40	.955	.927	.902	.877	.852	.828	.804	.80	.756	.733	.686	.639	.593	.546	.500	
42	.957	.931	.906	.882	.859	.835	.813	.790	.767	.745	.700	.655	.611	.566	.522	
44	.959	.933	.910	.887	.864	.842	.820	.799	.777	.755	.713	.670	.627	.585	.542	.500
46	.961	.936	.913	.891	.870	.849	.828	.807	.786	.765	.724	.683	.642	.602	.561	.520
48	.962	.939	.917	.896	.875	.855	.834	.814	.794	.774	.735	.695	.656	.617	.578	.539
50	.963	.941	.920	.900	.880	.860	.841	.821	.802	.783	.745	.707	.669	.631	.594	.556

NORMALIZED SCORES $E^*(k,t)$

(and Upper Bound $E_0(50,t)$)

Remaining Stockpile Size t

k	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20	22
1	.500															
2	.625	.500														
3	.695	.599	.500													
4	.742	.660	.581	.500												
5	.775	.705	.636	.569	.500											
6	.800	.738	.678	.619	.560	.500										
7	.820	.764	.711	.658	.606	.553	.500									
8	.836	.786	.737	.689	.642	.595	.548	.500								
9	.850	.803	.759	.715	.672	.629	.587	.544	.500							
10	.861	.818	.777	.737	.697	.658	.619	.580	.540	.500						
11	.871	.831	.792	.755	.718	.682	.646	.610	.573	.537						
12	.879	.812	.806	.771	.737	.703	.669	.635	.602	.568	.500					
13	.886	.851	.818	.785	.753	.721	.689	.658	.627	.595	.532					
14	.893	.860	.828	.797	.767	.737	.707	.678	.648	.619	.560	.500				
15	.899	.867	.837	.808	.779	.751	.723	.695	.667	.640	.584	.528				
16	.904	.874	.845	.818	.791	.764	.737	.711	.684	.658	.606	.553	.500			
17	.908	.880	.853	.827	.801	.775	.750	.725	.700	.675	.625	.576	.526			
18	.913	.886	.860	.835	.810	.786	.761	.738	.714	.690	.643	.596	.548	.500		
19	.916	.891	.866	.842	.818	.795	.772	.749	.726	.704	.659	.614	.569	.523		
20	.920	.895	.871	.848	.826	.804	.782	.760	.738	.716	.673	.630	.587	.544	.500	
22	.926	.903	.881	.860	.839	.819	.798	.778	.758	.738	.699	.659	.620	.580	.540	.500
24	.931	.910	.890	.870	.851	.832	.813	.794	.775	.757	.720	.684	.647	.611	.574	.537
26	.936	.916	.897	.879	.861	.843	.825	.808	.790	.773	.739	.705	.671	.637	.603	.569
28	.940	.921	.904	.886	.869	.853	.836	.820	.804	.787	.755	.723	.691	.660	.628	.596
30	.943	.926	.909	.893	.877	.861	.846	.830	.815	.800	.770	.739	.710	.680	.650	.620
32	.946	.930	.914	.899	.884	.869	.854	.840	.825	.811	.782	.754	.726	.697	.669	.641
34	.949	.934	.919	.904	.890	.876	.862	.848	.834	.821	.794	.767	.740	.713	.687	.660
36	.952	.937	.923	.909	.895	.882	.869	.856	.843	.830	.804	.778	.753	.728	.702	.677
38	.954	.940	.927	.913	.900	.887	.875	.862	.850	.838	.813	.789	.765	.740	.713	.692
40	.956	.943	.930	.917	.905	.893	.881	.869	.857	.845	.822	.798	.775	.752	.729	.706
42	.958	.945	.933	.921	.909	.897	.886	.874	.863	.852	.829	.807	.785	.763	.741	.719
44	.960	.947	.935	.924	.913	.901	.890	.879	.869	.858	.836	.815	.794	.773	.752	.731
46	.961	.949	.938	.927	.916	.905	.895	.884	.874	.863	.843	.822	.802	.782	.761	.741
48	.963	.951	.940	.930	.919	.909	.899	.889	.879	.869	.849	.829	.809	.790	.771	.751
50	.964	.953	.942	.932	.922	.912	.902	.893	.883	.873	.854	.835	.816	.798	.779	.760
E_0	.980	.971	.961	.951	.941	.931	.922	.912	.902	.892	.873	.853	.833	.813	.794	.774

function $f_d(x)$ also known to the defense. Suppose that the offensive weapon and decoy observations are presented one at a time to the defense, who also knows the total attack size n. If the defense has m missiles each of reliability ρ, how should they allocate to the sequentially-arriving offensive objects?

If the offensive weapon will certainly destroy the target whenever it is not successfully intercepted, one should allocate missiles to minimize the probability that the offensive weapon is not intercepted. It is clear that the specification of the optimum strategy is rather complicated:

> Suppose that there are $t \le m$ defensive missiles remaining in the stockpile, and $k \le n$ offensive objects yet to appear in the attack. When the first of the k offensive objects appears, note the value c of its observation, and allocate i defensive missiles to it if $c_i \le c \le c_{i+1}$.

Even this sort of strategy is applicable only if f_w and f_d satisfy a certain condition, as will be seen. Note that one can set c_0 equal to $-\infty$ and c_{t+1} equal to $+\infty$. This leaves t constants to be determined for a given $t \le m$, $k \le n$, or a total of $nm(m+1)/2$ constants for a strategy associated with m defensive missiles and n offensive objects.

The problem of finding an optimum strategy has been solved by Gorfinkel (1963a). He writes down the probability that the weapon will penetrate as

$$L(k,t) = \sum_{i=0}^{t} \frac{(1-\rho)^i}{n} \int_{c_i}^{c_{i+1}} f_w(x)\,dx + \sum_{i=0}^{t} \frac{(n-1)L(k-1,t-i)}{n} \int_{c_i}^{c_{i+1}} f_d(x)\,dx \quad .$$

If one differentiates $L(k,t)$ with respect to c_i, $i = 1,2,\ldots,m$, and sets these equal to zero, one obtains the following m implicit equations for determining the optimum values c_i^*:

$$f_w\left(c_i^*\right)/f_d\left(c_i^*\right) = (n-1)\Big(L(k-1,t-i) - L(k-1,t-i+1)\Big)/\rho(1-\rho)^{i-1} \quad .$$

Note that $f_w(x)/f_d(x)$ must be a monotone increasing function of x if these equations are to be uniquely solvable for the c_i. In statistical terminology, the probability density functions $f_w(x)$ and $f_d(x)$ are said to have a monotone likelihood ratio. If this condition is not satisfied, the specification of the optimum strategy takes a more complicated form than that given above.

Gorfinkel tabulated the L(k,t) associated with the optimum choice of the c_i for k = 1(1)20 and t = 1(1)10. The weapon and decoy observations were drawn from Gaussian probability density functions with standard deviations equal to unity. He considered four different cases:

Mean of Decoy Distribution	Mean of Weapon Distribution	Missile Reliability
0	1	0.8
0	2	0.8
0	1	0.999
0	2	0.999

In a subsequent handout, Gorfinkel (1963b) has postulated a more general model — the defense knows that exactly w weapons are present among the n offensive objects of the attack. He determined the optimum allocation strategies for two different criteria: minimizing the probability G(k,t,w) that one or more weapons are unintercepted, and minimizing the expected number E(k,t,w) of weapons which are unintercepted. Obviously the two criteria are identical when only one weapon is present; Gorfinkel showed (by example) that in general the two criteria lead to different allocation strategies. He presented tables of G(k,t,w) and E(k,t,w) for k = 1(1)15, t = 1(1)10, and w = 1,2,5. His weapon observations were drawn from a Gaussian probability density function with a mean and a variance of one; his decoy observations, from a Gaussian probability density function with a mean of zero and a variance of one. His defensive missile reliability, ρ, was assumed to be 0.9.

The missile allocation strategies presented in this section and the preceding one represent responses to quite different attacks. In the preceding section, it was assumed that the offense attacked with n weapons of equal (but unknown) destructive potential, and the threat of a weapon was measured by its miss-distance. In the present section, it was assumed that the offense attacked with exactly one (or exactly w) weapons of certain destructive potential and n - 1 (or n - w) decoys of no destructive potential, and the threat of an object was measured by a single real-valued number (by comparing it with the known probability density functions of weapons and decoys). The specification of an optimum missile allocation strategy thus depends critically upon what is assumed known by the defense about the attack. The reader is cautioned against using strategies under conditions for which they were not originally derived. In particular, if one is not certain whether the offense consists entirely of weapons with relatively small radii of destruction, or consists of a few weapons with relatively large radii of destruction mixed with decoys, then no reasonable strategy is yet available. It would clearly be desirable to derive "robust" strategies — that is, strategies which would work reasonably well against a wide range of possible offensive attacks, rather than optimally against a specific attack.

3.5 SHOOT-LOOK-SHOOT DEFENSE STRATEGIES

Thus far in this chapter, it has been assumed that the defense assigns missiles with individual reliabilities ρ in salvos to each offensive weapon. Obviously, the defense can improve performance if allowed to decide whether or not the ith missile of a salvo has destroyed the offensive weapon before having to commit the (i+1)st missile. If the ith missile did indeed destroy the weapon, then the (i+1)st missile can be used against other offensive weapons.

To fix ideas, a k-stage shoot-look-shoot defense strategy is defined as follows:

allocate m_1 missiles to n offensive weapons,

observe which weapons have been destroyed (say, n_1):

allocate m_2 missiles to the $n-n_1$ surviving weapons,

observe which weapons have been destroyed (say, n_2):

. . .

allocate m_{k-1} missiles to the $n-(n_1+n_2+\ldots+n_{k-2})$ surviving weapons,

observe which weapons have been destroyed (say, n_{k-1}):

allocate $m-(m_1+m_2+\ldots+m_{k-1})$ missiles to the $n-(n_1+n_2+\ldots+n_{k-1})$ surviving weapons.

This procedure will terminate earlier if all offensive weapons have been destroyed before the final look.

How should $(m_1,m_2,\ldots,m_{k-1})$ be selected? Obviously, one would like to determine the strategy for which the quantity

$$\sum_{i=0}^{n} (1-p)^i \Pr(\text{i offensive weapons survive})$$

is maximized. (As usual, p is the probability that a surviving weapon will destroy the target.) However, it is much easier to maximize the probability that no offensive weapons survive, and this is the criterion actually used. When p is equal to unity (any surviving weapon destroys the target), the two criteria are identical; it would be of interest to know under what broader conditions the equivalence holds. (If there are only two offensive weapons, the two-stage shoot-look-shoot strategy maximizing the probability of no survivors is equivalent to the shoot-look-shoot strategy minimizing the expected number of survivors.)

3.5.1 A Two-Stage Shoot-Look-Shoot Strategy

One can devise an algorithm for determining the optimum shoot-look-shoot strategy for any number of stages. By a set of recursion relations, the optimum k-stage shoot-look-shoot strategy is determined from the (k-1)-stage, (k-2)-stage,..., one-stage shoot-look-shoot strategies. However, these equations become so unwieldy that it is practical to derive only the two-stage shoot-look-shoot strategy.

To begin with, it is not hard to show that at each stage the allocated defensive missiles should be divided as evenly as possible among the surviving offensive weapons (as was mentioned at the start of the chapter). When n is equal to two, one can show that the allocation $(m_1, m-m_1)$ and the complementary allocation $(m-m_1, m_1)$ of defensive missiles to the first and second stages lead to the same probability that neither offensive weapon survives. Moreover, one can show that the optimum allocation is $m_1 = m/2$ for m even, and $m_1 = (m\pm1)/2$ for m odd. When n is equal to three or more, analytic results are more difficult to obtain, and a digital computer must be used to discover the optimum allocation $(m_1, m-m_1)$ for each m and n. There is a great deal of irregularity in the pattern of solutions — for example, if the missile reliability is $\rho = 0.8$ and there are six offensive weapons, then the optimum allocations for various m are:

m	10	11	12	13	14	15	16	17	18
m_1	6	6	6	7	6	6	12	11	12
$m-m_1$	4	5	6	6	8	9	4	6	6

The optimum allocation usually consists of an $(m-m_1)$ that is divisible by small integers; if only a few offensive weapons survive to the second stage, usually they can be attacked evenly by the remaining missiles.

The following approximate table ignores this fine structure and gives general guidelines for allocating missiles in a shoot-look-shoot strategy. Each entry in the table gives the set of defensive missile stockpiles corresponding to a specified attack size and first-stage allocation strategy. For example, assume that there are four offensive weapons, and that the defensive stockpile consists of 13 missiles. Since 13 lies in the interval 11 to 17 in the 4-weapon row, one concludes (by looking at the head of the column) that the optimum defense consists of engaging each attacker with two missiles in the first stage, and using the remaining five missiles against the survivors in the second stage. (Note that this differs slightly from the true optimum allocation of m_1 equal to 7 and m_2 equal to 6, given above.)

STOCKPILE OF DEFENSIVE MISSILES
REQUIRED FOR INDICATED STRATEGY

Offensive Weapons	Number of Defensive Missiles Allocated to Each Offensive Weapon at First Stage		
	1	2	3
2	2 to 5	6 to 9	10 to 13
3	3 to 8	9 to 13	14 to 18
4	4 to 10	11 to 17*	18* to 24
5	5 to 13	14 to 22†	
6	6 to 15		
7	7 to 17		
8	8 to 20*		
9	9 to 23*		
10	10 to 25*		

This table is valid for defensive missile reliabilities of 0.8, 0.9 and 0.95. For reliabilities of 0.5, all starred integers should be reduced by one (daggered, by two).

How good is the two-stage shoot-look-shoot strategy? One can compare the probability that no offensive weapons survive using an optimum two-stage strategy with the corresponding probabilities for a one-stage strategy and an m-stage strategy. The probability that no offensive weapons survive using a one-stage strategy is repeated from the beginning of this chapter:

$$P(1) = \left(1-(1-\rho)^k\right)^{n-r}\left(1-(1-\rho)^{k+1}\right)^r ,$$

where

$$k = [m/n] , \quad r = m - nk .$$

The m-stage strategy denotes that strategy which consists of assigning defensive missiles one at a time to offensive weapons until all offensive weapons are destroyed or all defensive missiles used up. The probability that no offensive weapons survive using an m-stage strategy is simply the probability that n or more defensive missiles do not fail:

$$P(m) = \sum_{i=n}^{m} \binom{m}{i} (1-\rho)^{m-i} \rho^{i} \ .$$

This can be evaluated with the aid of binomial probability tables.

For high values of missile reliability ρ, the two-stage shoot-look-shoot probability that no offensive weapons survive is much closer to the m-stage probability than the one-stage probability. In other words, providing a single "look" in a defensive strategy is quite worthwhile, but providing two or more "looks" is much less so unless the missile reliability is low. The following table illustrates the gains of a two-stage strategy. Denote the fractional gain of the two-stage strategy by the quantity $\left(P(2)-P(1)\right)/\left(P(m)-P(1)\right)$:

FRACTIONAL GAIN OF THE TWO-STAGE STRATEGY

Offensive Stockpile n	Defensive Stockpile m	Missile Reliability ρ 0.5	0.8	0.9	0.95
2	3	1.00	1.00	1.00	1.00
	4	1.00	1.00	1.00	1.00
	5	0.80	0.88	0.92	0.96
	6	0.75	0.86	0.90	0.95
4	6	1.00	1.00	1.00	1.00
	8	0.76	0.91	0.97	0.99
	10	0.70	0.94	0.99	1.00
	12	0.74	0.97	1.00	1.00
8	12	0.76	0.92	0.98	1.00
	16	0.54	0.93	0.99	1.00
	20	0.58	0.98	1.00	1.00
	24	0.71	1.00	1.00	1.00

3.5.2 Time-Limited Shoot-Look-Shoot Strategies

Consider now k-stage shoot-look-shoot strategies in which a single defensive missile is assigned to each offensive weapon at each stage and time is limited. In order for the defense to realize the potential benefits of a k-stage shoot-look-shoot strategy, offensive weapon arrival-times must be so widely separated that the defense has time to finish the shoot-look-shoot sequence against one offensive weapon before the next one arrives. However, this will not always be possible. The following idealized model of a fire-power-limited shoot-look-shoot defense has been adapted from Ordway and Rosenstock (1963).

Assume that a k-stage shoot-look-shoot strategy can be used against each offensive weapon in isolation. In other words, if T denotes the time-interval between the first possible assignment of a defensive missile to an offensive weapon and the destruction of the target by that weapon, and if τ denotes the time required for a single defensive missile to attack the weapon (including an evaluation of the outcome), then $k = \lfloor T/\tau \rfloor$. Assume that the offense attacks with n weapons arriving at equally spaced times t_o, $t_o + s\tau, \ldots, t_o + (n-1)s\tau$. Let the defensive missile reliability be ρ. Let the defensive missile stockpile be at least $(s(n-1)+k)$, so that there is no chance of running out of defensive missiles. If the defense uses the strategy "engage the nearest offensive weapon until it has been destroyed," what is the probability P_n that an offensive weapon will destroy the target?

For $s \geq k$, the problem is trivial: successive offensive weapon engagements are independent of each other and

$$P_n = 1 - \left(1-(1-\rho)^k\right)^n \quad .$$

However, when $s < k$, the evaluation of P_n is much more difficult. Assume that s is an integer, and let Q_i denote the probability that the ith offensive weapon is the first one to penetrate the defense. Then

$$P_n = \sum_{i=1}^{n} Q_i \quad .$$

Obviously, $Q_1 = (1-\rho)^k$ is the probability that the first offensive weapon is not destroyed. To evaluate Q_2, several possibilities must be considered. The probability that the first offensive weapon is destroyed on the ith trial, $1 \leq i \leq s$, and the second offensive weapon is not destroyed is given by

$$Q_2(i) = \rho(1-\rho)^{k+i-1} \; , \quad 1 \leq i \leq s \quad .$$

If the first offensive weapon is destroyed at trial (s+1) or later, then the engagement of the second offensive weapon is delayed. The defense can fire missiles on each of (s+k) possible trials altogether against the first and second weapons; thus

$$Q_2(i) = \rho(1-\rho)^{k+s-1} \; , \quad s+1 \leq i \leq k \quad .$$

Summing the $Q_2(i)$ from 1 to k, one obtains

$$Q_2 = (1-\rho)^k\left(1-(1-\rho)^s + (k-s)\rho(1-\rho)^{s-1}\right) \quad .$$

Unfortunately, computation of higher terms is quite tedious; for example, in calculating Q_3 one must consider the two cases $2s < k$ and $2s \geq k$ separately. Ordway and Rosenstock found it necessary to use a digital computer to compute P_n.

The effectiveness of the defense can be characterized by the average number of offensive weapons destroyed by the defense before the first penetrator (assuming infinite offensive and defensive stockpiles). This is given by

$$E = \sum_{i=i}^{\infty} iQ_{i+1} \quad .$$

(E+1) can be approximated by the median — that $n = n_o$ which is a solution to the equation

$$0.5 = \sum_{i=1}^{n} Q_i = P_n \quad .$$

The probability of target destruction, P_n, and the average number of offensive weapons to the first penetrator, E, can be readily calculated for two special cases. First, let $s = 1$ (the time-interval between successive arrivals of offensive weapons is equal to the time required for a single missile to attack a weapon). The probability of target destruction is the probability of having (n-1) or fewer successful defensive missile firings prior to the occurrence of the kth defensive missile failure. This is given by the cumulative negative binomial distribution,

$$P_n = \sum_{i=0}^{n-1} \binom{k+i-1}{i}(1-\rho)^k\rho^i \quad .$$

Note that the first two terms of this sum are equal to Q_1 and Q_2 with $s = 1$. The average number of weapons destroyed is given by $E \sim k\rho/(1-\rho)$ when n goes to infinity.

For the second special case, let $s = 0$ (the offensive missiles all arrive at the same time). The probability of target destruction is equal to the probability of having $n - 1$ or fewer successful defensive missile firings in k trials. If $n > k$, this probability is obviously unity; if $n < k$, it is given by the cumulative binomial distribution

$$P_n = \sum_{i=0}^{n-1} \binom{k}{i} (1-\rho)^{k-i} \rho^i \quad .$$

Note that the first two terms of this sum are equal to Q_1 and Q_2 with $s = 0$. The average number of weapons destroyed is given by $E = k\rho$ for any $n \geq k$.

Ordway and Rosenstock conjecture for arbitrary values of s that E, the average number of weapons to the first penetrator, can be approximated by the expression

$$E' = k\rho/(1-s\rho) \quad ,$$

which agrees with E when $s = 0$ and $s = 1$. This approximation is likely to be satisfactory only for $0 \leq s \leq 1$; note that E' is infinite when $s = 1/\rho$. For a number of different choices of (ρ,k,s), Ordway and Rosenstock used a digital computer to calculate the median n_0 and compared this quantity with E'. In most cases, they found that n_0 and E' agreed within a few per cent.

The simultaneous attack ($s = 0$) has also been analyzed by Morgenthaler (1960); however, his model is a somewhat more detailed one. He breaks up the defensive missile attack time τ into several components — a time τ_1 to send up a defensive missile, a time τ_2 to evaluate the results of an engagement, and a time τ_3 to switch the defense from one offensive weapon to another (if the engagement is a success). Note that in the model used by Ordway and Rosenstock, $\tau = \tau_1 + \tau_2$ and $\tau_3 = 0$. Morgenthaler determines the probability P(j) that j offensive weapons have been destroyed by the time the first offensive weapon penetrates the defense, as well as the probability Q(N) that N defensive missiles have been used. Obviously, his P(j) is analogous to the probability Q_{j+1} defined earlier; the reader interested in formulas for P(j) and Q(N) (which are somewhat complicated) is referred to his paper.

3.6 DEFENSES LIMITED BY TRAFFIC-HANDLING CAPABILITY

In the preceding section, the defense of a single point target was overwhelmed whenever offensive weapons arrived too rapidly for a shoot-look-shoot strategy to absorb them. However, if traffic-handling capability is considered, this problem need not be confined to shoot-look-shoot strategies. Suppose that each weapon is intercepted by at most one missile. The defense of a target is ordinarily implemented with a radar which observes the flight paths of the incoming weapon and the missile assigned to intercept it. It may happen that the final T seconds before intercept requires the undivided attention of the radar; it is not available for monitoring other offensive weapon interceptions. If too many weapons arrive at once, the interception of one or more of them may be so much delayed that they damage the target first.

The following elementary model of traffic-handling capability may provide some insight. Assume that offensive weapons are launched in such a manner that they reach the target at equally spaced arrival times t_0, $t_0 + s$, $t_0 + 2s, \ldots$. However, the weapons are independently subject to launch failures, so that each weapon actually has a probability p of arriving at the target. Each arriving weapon is assigned a single defensive missile which has a probability ρ of destroying the weapon. The offensive weapons penetrate the defense for either of two reasons:

(1) defensive missile failure (with probability $1 - \rho$), or

(2) traffic-handling failure (r consecutive offensive weapons are launched without failure).

In many circumstances, a reasonable criterion of effectiveness is the expected number of penetrators in n weapon launches. One such situation is that in which the target is relatively impervious to damage (that is, if the probability that a penetrating weapon destroys it is low). In order to simplify the mathematics, overlapping strings of successful launches causing traffic-handling failure are not allowed; for example if r = 3, then a string of 4 or 5 consecutive successful launches is counted as only one pentration, but a string of 6 consecutive successful launches is counted as two penetrations.

One can show that the expected number of penetrators in n weapon launches can be put in the following form:

$$E = n\left(p(1-\rho)+(1-p)A\right)+ A\left(1-(1-p)r/(1-p^r)\right)$$
$$+ Ap^{tr}\left((1-p)r-1+(1-p)rA/\rho\right) - As(1-p)p^{tr} \quad ,$$

where $A = p^r \rho/(1-p^r)$ and $n = tr + s$, $0 \leq s < r$. Ordinarily, the first term (a linear function of n) plus the second term (a constant independent of n) will yield adequate accuracy. The first term gives an upper bound for E.

It is worth noting that this model can give some insight into shoot-look-shoot strategies as well, by exchanging the roles of offense and defense. Assume that an offensive weapon penetrates if all defensive missiles in an r-stage shoot-look-shoot fail. If one replaces p with $1 - \rho$ in the above formula and then replaces ρ by 1, the resulting formula gives the expected number of penetrators after n defensive missiles have been used against a set of offensive weapons arriving sequentially.

The principal drawback of the above traffic-handling model, as well as the shoot-look-shoot models discussed in the previous section, is the unrealistic assumption concerning the offensive attack. It would be much more realistic to consider an attack not of equally spaced arrival times, but of arrival times consisting of n observations drawn at random from (say) a Gaussian distribution function and arranged in order from earliest to latest. However, this leads to analytic problem of considerable difficulty. Suppose one simplifies the model by assuming defensive missiles of perfect reliability ($\rho = 1$), thereby eliminating the necessity of using shoot-look-shoot strategies. Let and standard deviation of the probability density function of offensive weapon arrival times be denoted by σ, and (as before) let the time required for a single defensive missile to engage an offensive weapon be denoted by T. If offensive weapons can be engaged only one at a time in the order of their arrival, then certain offensive weapons cannot be engaged at the time of their arrival because the defense is still occupied with earlier weapons. What is the probability density function of the maximum engagement delay of any of the offensive weapons? Can this be expressed in closed form as a function of n, σ and T?

When n = 2, one can show that the probability that the second weapon will encounter a delay between t and t + dt is given by

$$p(t)dt = (1/\sigma\pi^{1/2})\exp\left(-(t-T)^2/4\sigma^2\right)dt , \qquad \text{for } t > 0 \quad .$$

The probability of no delay is obtained by integrating this expression with respect to t from $-\infty$ to zero. For n equal to 5, 10, 20 and 40, it is necessary to resort to Monte Carlo simulation on a computer to obtain the expected value of the normalized maximum delay (dividing the delay by σ). The results below are accurate to about 10 per cent, except for n = 2, where they have been calculated by means of the preceding formula.

EXPECTED VALUE OF NORMALIZED MAXIMUM DELAY t/σ

Offensive Stockpile	Normalized Total Engagement Time nT/σ		
n	1	2	4
2	0.104	0.396	1.33
5	0.09	0.35	1.2
10	0.08	0.30	1.1
20	0.06	0.22	1.0
40	0.035	0.15	0.9

If one can assume that the n arrival-times are drawn from an exponential distribution function instead of a Gaussian one, then it is possible to obtain an analytic expression for the probability of no delay. Adapting an argument due to Kabak (1972), one obtains

$$\Pr(\text{no delay}) = a^{n-1}(n-1)!\,\exp(-aTn(n-1)/2) \quad ,$$

where the exponential probability density function is given by $f(t) = a\exp(-at)$. Kabak actually obtains a more general result: he assumes that the radar processing time T is not a constant, but a random value from the gamma distribution $g(t) = b^j t^{j-1}\exp(-bt)/(j-1)!$. In this situation,

$$\Pr(\text{no delay}) = \prod_{i=1}^{n-1} (1+ia/b)^{-j} \quad .$$

However, the expected value of the maximum delay must (as before) be obtained by simulation.

3.7 SUMMARY

This chapter presents a wide variety of mathematical models in which the structure of the target does not come into play. In general, these models can be considered to represent the simplest attack-defense situation — a single target defended by a stockpile of identical unreliable missiles.

If the defense knows the cookie-cutter damage radius of a salvo of attacking weapons, defense allocation is easy; if the damage radius is not known, it is possible to maximize the expected distance to the nearest unsuccessfully-intercepted weapon. If the attack is sequential and the damage radius is known, a variety of models is possible. In one, the attack size is completely unknown and the defense maximizes the expected number of weapons successfully intercepted prior to the first penetrator; in a second, the probability density function of the attack size is known and the expected number of penetrators is minimized; in a third, defense missiles are assigned so that the probability of target destruction is proportional to attack size (up to a limit determined by the missile stockpile).

If the attack is sequential and the damage radius is unknown, then it is necessary to assume that the attack size is known in order to derive a strategy for deciding whether later weapons will impact closer than the present weapon. If the aiming-error of the attacker is also unknown, this can be inferred from early weapon-arrivals.

The latter part of the chapter considers the additional options open to the defense if he can evaluate the success or failure of a missile in time to launch a subsequent missile against the same weapon. On the other hand, the defense may be fire-power limited; that is, he may be unable to engage all weapons because each weapon requires a finite amount of processing time by the defense system. If weapons arrive at equally-spaced times, the expected number of weapons successfully intercepted before the first penetrator can be calculated; if weapons arrive with an exponential or Gaussian dispersion in time, the problem appears to be analytically intractable.

CHAPTER FOUR

OFFENSE AND DEFENSE STRATEGIES FOR A GROUP OF IDENTICAL TARGETS

Preceding chapters have focused upon the probability of survival in the no-defense case, and the optimum defense strategies to be used for a target of unspecified structure when certain characteristics of the attack are not known. In contrast, this chapter considers one simple type of target structure; it specifies useful offense and defense strategies when a group of identical targets is under attack. Both offense and defense strategies depend upon the degree of knowledge each side has concerning the other's stockpile size and allocation to individual targets: the main purpose of this chapter is to show how this knowledge affects the choice of a strategy. Strategy is also affected by self-knowledge. For example, if the defense can determine which targets have been destroyed early in the attack, he can allocate defensive missiles to undamaged targets; this is known as damage assessment. As another example, if the defense cannot determine which target an offensive weapon is directed against in time to make an intercept if desired, the defense strategy is also modified. The ability to identify the target being attacked is known as attack evaluation.

How does one specify an allocation of offensive weapons or of defensive missiles? Typically, the offensive weapon allocation is straightforward; it consists simply of the assignment of a specified number of weapons to each target in the group. However, the allocation of defensive missiles need not be this simple. One can assign a specified number of missiles to the defense of each individual target; or, one can assign a specified number of defensive missiles to a subgroup of targets. In the latter case, defensive missiles are used against any offensive weapon directed against any target in the subgroup. In the limit, one can elect to defend the entire subgroup of targets with the entire defensive missile stockpile. In general, it is much easier to analyze defensive strategies which assign missiles to individual targets; the outcomes of these engagements do not depend upon the order of arrival of offensive weapons. Strategies allocating weapons and missiles to individual targets are called preallocation strategies, and will be considered in the first part of this chapter.

4.1 PRELIMINARIES CONCERNING PREALLOCATION STRATEGIES

Sections 4.1 through 4.3 discuss the properties of preallocation strategies. The present section concerns itself with the basic ideas behind preallocation strategies, and will lay out the assumptions and notation that will be used. However, some of the assumptions and much of the notation will also be employed in later sections dealing with other types of strategy.

Some comments on the rationale behind preallocation strategies are in order here. The fact that such strategies are easier to analyze has already been noted. Despite the fact that preallocation strategies are notationally formidable, they represent effectively computable exact solutions of reasonably realistic problems; this is in general not true for most of the later results in this chapter.

A more important advantage of preallocation strategies, however, is that in many cases they are more effective for the defense than other strategies. Suppose that the defense uses the simple strategy of firing at each arriving weapon until his stockpile is exhausted. Suppose that the offensive stockpile substantially outnumbers that of the defense, a common situation. Then the offense can first send in a force of weapons equal to the defensive stockpile, and then attack the targets with the knowledge that they will not be defended any more. The use of a preallocation defense precludes this. In Section 4.4, a number of defenses will be considered which are intermediate between the above simple strategy and a preallocation strategy. In general, for the values of the parameters selected, these perform worse than the best preallocation strategy.

However, the reader is cautioned against assuming that preallocation defenses are always best. If the defense equals or outnumbers the offense, and the defensive missile reliability is reasonably high, the simple strategy given above will clearly outperform any preallocation strategy.

All targets are assumed to be identical (that is, each target has the same value to the offense and defense); this assumption is removed in the next chapter. The targets are located sufficiently far apart so that an offensive weapon which destroys one target does not affect any other target.

With brief exceptions (which will be noted when they occur), all missile engagements are one-on-one; that is, at most one defensive missile is assigned to each offensive weapon. This assumption is justified if the defensive missile reliability is quite high.

The criterion of effectiveness is E(f), the expected fraction of targets saved; the defense selects strategies which maximize this quantity, and the offense selects strategies which minimize this quantity. If a target is defended by i defensive missiles and has j offensive weapons preallocated to it, the probability of target survival is given by

$$\Pr(\text{target survives}) = q_1^{\min(i,j)}\, q_0^{\max(0,j-i)} \quad ,$$

where

q_1 = Pr(target survives attack by single intercepted weapon)
$= 1 - p + \rho p = 1 - p(1-\rho)$,

q_0 = Pr(target survives attack by single unintercepted weapon)
$= 1 - p$.

It is assumed that both the offense and the defense know the target kill probability p and the defensive missile reliability (kill probability) . This implies that the offense knows the hardness of each target (assumed to be the same for all), and the defense knows the offensive weapon yield and accuracy (assumed to be the same for all).

The expected fraction of targets saved, E(f), is given by the summation of Pr(target survives) over all the targets.

The number of offensive weapons available per target is denoted by a, and the number of defensive missiles available per target is denoted by d. Each side is assumed to know the value of both a and d. The validity of the assumptions about a may be doubtful if there are two or more regions containing targets, each region having its own defense. Although it appears plausible that the offense can know the defense stockpiles, it is not at all clear in practice that the defense will know what fraction of the (possibly much larger) offense stockpile will be assigned to a given group of targets.

This assumption may be somewhat more palatable if recast in the following form. It is possible that the defense can specify a "maximum tolerable damage level" to the group of targets — that is, lower damage levels can be endured but higher levels cannot. The defense then will plan his defensive strategy under the assumption that the enemy has a stockpile just large enough to impose the maximum tolerable damage level upon the defense if both use optimum strategies. If, in reality, the enemy has a larger stockpile, the defensive strategy will not be optimum and the enemy will impose a larger-than-necessary damage upon the defense; but this will not matter, because the targets would have been intolerably damaged no matter what the defense could do. On the other hand, if the enemy has a smaller stockpile, the defensive strategy will again be nonoptimum and the enemy will again impose a larger-than-necessary damage upon the defense; but this damage level will still be a tolerable one.

Another way of getting around the problem of defensive knowledge of the enemy stockpile is to reformulate the criterion of effectiveness. The defense can design a strategy such that his expected loss is insensitive to enemy attack size; that is, such that

$$1 - E(f) = ka \quad .$$

Of course, the value of k will depend upon the number of defensive missiles available. Beyond a certain value of a, the expected fraction of targets destroyed will fall below ka, and as a goes to infinity the expected fraction will approach unity.

If T is not too small, it is reasonable to specify the offense and defense strategies in terms of strategy levels involving continuous variables. Offensive strategy levels are specified by the vector $y = (y_0, y_1, y_2, \ldots)$, where y_i denotes the fraction of the targets in the group which are assigned i offensive weapons each. Similarly, defensive strategy levels are specified by the vector $x = (x_0, x_1, x_2, \ldots)$. Note that the x_i and y_i are constrained by the equations

$$1 = \sum_i x_i \ , \qquad 1 = \sum_i y_i \ ;$$

$$d = \sum_i ix_i \ , \qquad a = \sum_i iy_i \ .$$

Those preallocation strategies may not be realizable because of the finite number of targets in the group. If T targets are present, the x_i and y_i can be reasonably approximated by fractions of the form k/T, k = 0,...,T. However, if T is very small (say, ten or less), it may be advisable to work out the offensive and defensive strategy levels by formulating the problem as a matrix game and solving it by the usual techniques of game theory. Throughout most of this chapter, the difficulty of realizing strategies based on continuous variables in terms of integers will be ignored.

4.2 OFFENSE-LAST-MOVE AND DEFENSE-LAST-MOVE STRATEGIES

Assume that preallocation strategies are to be used, and that the defense can carry out attack evaluation but not damage assessment. Assume also that both the defense and the offense know a, d, q_0 and q_1. To simplify the problem, assume that a and d are both integers. One can readily derive upper and lower bounds for the expected fraction of targets saved, assuming that the offense and defense both act rationally. The lower bound is achieved when the offense can see the defensive allocation of missiles to targets before making his own allocation (defense-last-move). These bounds can be expressed mathematically as follows:

$$\text{Upper bound:} \quad \min_{y} \max_{x} E(f)$$

$$\text{Lower bound:} \quad \max_{x} \min_{y} E(f)$$

where the maxima and minima are taken over all possible defense and offense preallocation strategies. It should be noted that if one side has the last move, the distinction between preallocation and nonpreallocation strategies for that side is lost.

If the offense can see the defense allocation before choosing his own, it is not difficult to determine that the best possible defense strategy is to allocate an equal number of missiles to each target ($x_d = 1$). What is the optimal offense strategy against this defense? Let the offense attack a fraction of the targets $y_k = a/k$ with k weapons apiece, $k \geq a$, and let $y_0 = 1 - (k/a)$: that is, the remaining targets are unattacked. Let P(k) denote the probability of destruction of a target attacked by k weapons and defended by d missiles. The average return per weapon at an attacked target can be defined as $\lambda = P(k)/k$, and the expected fraction of targets destroyed is $1 - E(f) = y_k P(k) = a\lambda$. Assume that P(k) is a function for which a unique value of k, denoted by k*, maximizes λ:

$$\lambda^* = \max_{k} \lambda = \frac{P(k^*)}{k^*} \quad .$$

It is obvious that if $k^* > a$, the offense allocation which maximizes λ also maximizes 1 - E(f). Otherwise, $y_a = 1$ (attack all targets with a weapons apiece) is the optimal offense strategy. In short, $1 - E(f) = \lambda^* a$, $0 < a \leq k^*$; $1 - E(f) = P(a)$, $k^* \leq a$.

For the one-on-one defense assumed generally throughout this chapter,

$$P(k) = q_1^{\min(d,k)} \, q_0^{\max(0,k-d)} \quad .$$

However, the argument presented above can be extended to other situations, for example salvo defenses. A few practical comments may be in order here. The value k* can be found graphically by constructing a tangent to P(k) passing through the origin; that is, λ is the slope of a line passing through (0,0) and (k,P(k)). The maximum λ corresponds to that value of k closest to the tangent.

Note that if $P(k)$ is continuous, then at the tangent point the marginal and average returns are equal: $dP(k)/dk = P/k$.

So far an attack against targets in a single module has been considered. This technique can be extended to allocations of weapons to non-overlapping modules of targets by making $P(k)$ the expected fraction of a module that is destroyed if k weapons are allocated to the module, and the offense and defense strategies within the module are known. It is useful for defense design purposes because given a tolerable level of destruction, E^*, and offense stockpile per module, a, the defense designer need only assure that the function $P(k)$ does not pass through a line from the origin with slope E^*/a. This is, however, only a sufficient, not necessary, condition: a more general condition is that $P(k)$ cannot be more than tangent to any line through (a,E^*) with slope less than or equal to E^*/a. Otherwise, the offense can achieve a payoff per weapon greater than E^*/a by an attack at two (or more) levels.

Another sort of offense-last-move case will be considered, where the model is an "attacker-oriented" one.

If the defense can see the offense allocation before choosing his own, it is not difficult to determine that the best possible offense strategy is to allocate an equal number of weapons to each target ($y_a = 1$). If $d \geq a$, the maximizing defense strategy is obvious: attack each weapon with a single missile. (Only one-on-one engagements are permitted in this section.) Therefore, it is necessary to derive the defense strategy only for $d < a$. It is not difficult to show that the maximizing defense strategy is $(x_0 = (a-d)/a,\ x_a = d/a)$ and the corresponding expected fraction of targets saved is

$$E(f) = \frac{a-d}{a} q_0^a + \frac{d}{a} q_1^a \quad .$$

Assume that the attack size, a, is not known to the defense. If the offense has the last move, how should the defense be allocated so that $1 - E(f) = ka$ for $0 \leq a \leq a_0$? If $q_1 = 1$ and $q_0 = 0$, the answer is easy: allocate the same number of missiles to each target. The offense will attack targets with $(d+1)$ weapons apiece until his stockpile is exhausted; $1 - E(f) = a/(d+1)$, $0 \leq a \leq d+1$. However, for arbitrary q_1 and q_0 the answer to this problem is not known.

4.3 STRATEGIES WHEN NEITHER SIDE KNOWS THE OTHER'S ALLOCATION

Assume, as before, that preallocation strategies are to be used, and that the defense can carry out attack evaluation but not damage assessment. Assume also that both the offense and the defense know a, d, q_0 and q_1. Assume that the offense allocates its weapons and the defense allocates his missiles, each in ignorance of the other's allocation. What allocations should each side select, and what is the expected fraction of targets saved corresponding to these allocations?

This allocation problem can be formulated in terms of a zero-sum two-person game. The payoff is the fraction of targets saved, and the strategies of the two players consist of (1) a specification of the fractions of targets to be assigned given offense levels $(y_0, y_1, y_2, \ldots)$ and defense levels $(x_0, x_1, x_2, \ldots)$, satisfying the constraints of Section 4.1, and (2) an assignment of individual targets to these offense offense and defense levels. It is easily seen that only a finite number of offense and defense levels are reasonable. Then a generalization, due to Charnes (1953), of the fundamental theorem of game theory states that the game has a value, V, and that there exist optimum probability density functions of defense and offense strategies. The optimum defense strategy has the property that, if the defense selects a strategy according to this probability density function, then the offense cannot produce an expected outcome less than V, no matter what strategy he selects. Similarly, the optimum offense strategy has the property that, if the offense selects a strategy according to this probability density function, then the defense cannot achieve an expected payoff greater than V, no matter what strategy he selects. In short, the defense can select a strategy which guarantees that the expected fraction of targets saved is at least V, and the offense can select a strategy which guarantees that the expected fraction of targets saved is at most V, the value of the game. One can further prove that

$$\max_x \min_y E(f) = V = \min_y \max_x E(f) \quad ,$$

as might be expected.

The probability density function of offense strategies which guarantees an expected payoff of at most V can be expressed in a simple form: select a fraction y_0 of the targets at random for no attack, select a fraction y_1 of the targets at random for attack by one weapon, and so on. The probability density function of defense strategies which guarantees an expected payoff of at least V can be similarly expressed. In other words, the solution to the allocation problem consists of finding the vectors $(y_0, y_1, y_2, \ldots)$ and $(x_0, x_1, x_2, \ldots)$; the expected fraction of targets saved is then

$$V = \sum_{i,j} x_i y_j q_1^{\min(i,j)} \, q_0^{\max(0,j-i)} \quad .$$

4.3.1 An Explicit Solution to the Preallocation Problem

This problem can be expressed as a constrained game; such games can be solved by linear programming, as has been shown by Charnes (1953). The possibility of using linear programming will be discussed in Section 4.3.4. However, Matheson (1967) has found a solution to the preallocation problem which does not use linear programming explicitly.

The results of Matheson's work are summarized in the following paragraphs. Unfortunately, the solution is relatively difficult to characterize in a concise form: those wishing to use it will find a digital computer almost indispensable.

Matheson proves that the solution can be based on partitioning the positive quadrant of the (a,d)-plane into regions that are rectangular or are infinite rectangular strips with boundaries dependent on p and ρ, or, equivalently, on q_0 and q_1. The regions are of four types, each having a special form of offense and defense strategy associated with it. Thus, the first problem is to identify in what region (a,d) is located. This is accomplished by means of the table on the next page giving the boundary-values of the various regions. These boundary-values are denoted by terms of the form $D_i(x,y)/T$ and $A(x,y)/T$, respectively, and are termed critical defense and critical offense levels by Matheson. (The reader is warned that Matheson's notation has been considerably altered.) The various critical values are defined below:

$$D_1(m-1,h)/T = \frac{(m-h)e(m,h)}{\rho q_1^{h-1}} \left\{ h - 1 + q_1^{-m+h+1} \frac{1-q_1^m}{1-q_1} \right\} ,$$

$$D_2(h+w,h)/T = (h-1) + D_1(1+w,1)/T ,$$

$$D_3(h+w,h)/T = h + D_1(w,1)/T ,$$

$$A(m-1,h)/T = h + \frac{b(m,h)(m-h-1)}{q_1^{m-2}} \left\{ \frac{1}{q_1-q_0} + \frac{1-q_0^{m-h-2}}{1-q_1} \right\} ,$$

where

$$e(m,h) = \frac{\rho q_0^{h-1}}{\rho/p + (m-h)\left(\rho - 1 + (q_0/q_1)^{h-1}\right)} ,$$

BOUNDARY-VALUES OF RECTANGULAR REGIONS

Rectangle Type	Critical Defense Levels	Critical Offense Levels
I	$D_1(m-2,h)/T < d < D_1(m-1,h)/T$ $D_1(m-2,h)/T < d < D_2(m-1,h)/T$	$A(m-1,h-1)/T < a < A(m-1,h)/T$
II	$D_1(m-1,h+1)/T < d < D_1(m-1,h)/T$	$A(m-1,h)/T < a < A(m,h)/T$
III	$D_2(h+w,h)/T < d < D_3(h+w,h)/T$ $D_3(h+w,h)/T < d < D_2(h+w+1,h+1)/T$	$A(h,h)/T < a < A(h+w+1,h)/T$
IV	$D_2(h+w,h)/T < d$	$0 < a < A(h,h)/T$

$$b(m,h) = \frac{1}{\dfrac{h}{\left(q_1^h - q_0^h\right)} + \left(\dfrac{(1-q_0)\left(1-q_1^{m-h-1}\right)}{(q_1-q_0)(1-q_1)}\, q_1^{m-2}\right)} \quad ,$$

and w is the largest integer less than $\rho/p(1-\rho)$.

Once the region in which (a,d) is located has been determined from the above relationships, the values of $m-1$ and h are uniquely specified. It is then necessary to determine corresponding values of ℓ and g using the table below:

Region Type		
I	$m-1 = \ell$,	$h-1 = g$
II	$m = \ell$,	$h = g$
III	$m-1 = \ell$,	$h = g$
IV	$\ell \leq h$	

Knowing ℓ and g, one can now specify the offense and defense strategies and the expected fraction of targets saved. The offense and defense strategies are obtained by linear interpolation between critical strategies. The appropriate critical strategies are determined by noting the form of the critical defense and offense levels bounding the region in which (a,d) lies — these levels, it will be recalled, are of the form $D_1(x,y)/T$, $D_2(x,y)/T$, $D_3(x,y)/T$ and $A(x,y)/T$.

Critical Defense Strategy Corresponding to $D_1(m-1,h)/T$:

$$x_0 = 1 - \frac{(m-h)e(m,h)}{\rho q_1^{h-1}} \quad ,$$

$$x_i = \frac{e(m,h)}{\rho}\left(\frac{m-i}{q_1^{i-1}} - \frac{m-i-1}{q_1^{i}}\right) \quad \text{for} \quad i = h,\ldots,m-1 \quad .$$

Critical Defense Strategy Corresponding to $D_2(h+w,h)/T$:

$$x_h = 1 - \frac{1 - p(1-\rho)/\rho}{q_1}$$

$$x_i = \frac{1-(i-h)p(1-\rho)/\rho}{q_1^{i-h}} - \frac{1-(i-h+1)p(1-\rho)/\rho}{q_1^{i-h+1}} \quad \text{for} \quad i = h+1,\ldots,h+w-1 \quad .$$

Critical Defense Strategy Corresponding to $D_3(h+w,h)/T$:

$$x_h = 1 - \frac{w}{\rho(w+1/p)} \quad ,$$

$$x_i = \frac{1}{\rho(w+1/p)}\left(\frac{w-i+h+1}{q_1^{i-h-1}} - \frac{w-i+h}{q_1^{i-h}}\right) \quad \text{for} \quad i = h+1,\ldots,h+w-1 \quad .$$

Critical Offense Strategy Corresponding to $A(\ell,g)$:

$$y_g = 1 - \frac{b(\ell,g)}{q_1^{\ell-1}}\left\{\frac{1}{(q_1-q_0)} + \frac{1-q_1^{\ell-g-1}}{1-q_1}\right\} \quad \text{for} \quad i = g+1,\ldots,\ell-1 \quad ,$$

$$y_i = \frac{b(\ell,g)}{q_1^{i}} \quad ,$$

$$y_\ell = \frac{b(\ell,g)}{(q_1-q_0)q_1^{\ell-1}} \quad .$$

These strategies can be substituted into the equation for V, the expected fraction of targets saved; the result is algebraically cumbersome and is omitted here.

4.3.2 A Simplified Problem: Perfect Offensive Weapons and Defensive Missiles

If q_1 is set equal to 1 and q_0 is set equal to 0, the offense and defense strategies are much simpler to specify. Following Matheson (1966), it is possible to write down the offense and defense strategies directly as functions of a and d, instead of the auxiliary quantities g, h, ℓ, and m. Instead of four regions in the (a,d)-plane corresponding to rectangle types I, II, III and IV, there are only two:

$$\text{Offense dominant (Type II)} \quad \text{if} \quad [2d+1] < [2a] \quad ,$$

$$\text{Defense dominant (Type I)} \quad \text{if} \quad [2d+1] \geqq [2a] \quad .$$

If the defense is dominant, then

Defense Strategy:

$$x_i = \frac{2([2d+1]-d)}{[2d+2][2d+1]} \quad \text{for} \quad i = 0,1,2,\ldots,[2d] \quad ,$$

$$x_{[2d+1]} = \frac{2d - [2d]}{[2d+2]} \quad .$$

Offense Strategy:

$$y_i = \frac{2a}{[2d+1][2d+2]} \quad \text{for} \quad i = 1,2,\ldots,[2d+1] \quad ,$$

$$y_0 = 1 - \frac{2a}{[2d+2]} \quad .$$

The expected fraction of targets saved is

$$V = 1 - \frac{2a([2d+1]-d)}{[2d+1][2d+2]} \quad .$$

when $\underline{d}$ is an integer, the expected fraction of targets saved is equal to $1 - a/(1+2d)$. If the offense is dominant, then

Defense Strategy:

$$x_i = \frac{2d}{[2a][2a-1]} \quad \text{for} \quad i = 1,2,\ldots,[2a-1] \quad ,$$

$$x_0 = 1 - \frac{2d}{[2a]} \quad .$$

Offense Strategy:

$$y_i = \frac{2([2a]-a)}{[2a][2a-1]} \quad \text{for} \quad i = 1,2,\ldots,[2a-1] \quad ,$$

$$y_{[2a]} = \frac{2a - [2a]}{[2a]} \quad .$$

The expected fraction of targets saved is

$$V = \frac{2d}{[2a-1]}\left(1 - \frac{a}{[2a]}\right) \quad .$$

When a is an integer, the expected fraction of targets saved is equal to $d/(2a-1)$.

Assume that the attack size, a, is not known to the defense; how should the defense allocate his missiles so that the expected fraction of targets lost is proportional to the actual attack size? This criterion of effectiveness can be easily achieved for certain defense levels d: find that (d,a) pair corresponding to (m,1), where m is an integer. In effect, one assumes a certain offense level and defends in an optimum way against this offense. When q_1 is equal to one and q_0 equal to zero, the defense and offense strategies corresponding to an offense-dominant attack satisfy the criterion:

$$E(\text{fraction of targets lost}) = \frac{2a([2d+1]-d)}{[2d+1][2d+2]} \ .$$

4.3.3 Arriving at Integral Allocations

The preceding two sections have specified the Matheson offense and defense strategies in closed form. It is important to realize that the specification of the Matheson strategies, from the standpoint of actually assigning missiles (or weapons) to targets, is not complete. All that the Matheson defense strategy does is specify that a randomly chosen fraction x_0 of the targets will be assigned zero missiles, a randomly chosen fraction x_1 of the targets will be assigned one missile, and so on. Since the number of targets, T, is finite, there is no guarantee that these fractions can be achieved by any real assignments. As a working rule, one can select fractions as close as possible to $(x_0,x_1,\ldots,x_{m-1})$ and $(y_0,y_1,\ldots,y_m)$ and assume that the expected fraction of targets is well-approximated by the Matheson game value E(f). In Section 4.3.5 it will be seen that this assumption is reasonable.

However, one can sometimes do more than this. To be specific, one can define an integer strategy game analogous to the Matheson game, in which the mixed strategy used by the defense (or the offense) is a probability distribution function $(p_1, p_2, \ldots,p_N)$ taken over the N different pure strategies (the actual assignment of an integral number of missiles to each of the T targets). This integer strategy game is impossible to solve in closed form except for extremely small numbers of missiles, weapons and targets because the number of pure strategies (that is, the number of ways D (or A) objects can be partitioned among T cells) quickly becomes very large. Matheson was unable to obtain a general formula for the strategies in terms of A, D, T, q_0 and q_1; instead, as has been seen, he defined the Matheson game by enlarging the space of mixed strategies to include <u>all</u> strategies of the form $(x_0,x_1,\ldots,x_{m-1})$ realizable under stockpile constraints, not just those realizable from the N pure strategies.

One can prove that for perfect offensive weapons and defensive missiles ($q_0 = 0, q_1 = 1$), the value of the Matheson game is the same as the value of the integer strategy game. (In other words, that the optimum can always be realized in terms of the N pure strategies.) This result does not hold for imperfect weapons and missiles, since it is possible to construct examples in which it is impossible to find a mixed strategy in the integer strategy game satisfying the requirements of the optimum strategy in the Matheson game. However, the number of pure strategies (and therefore the variety of mixed strategies) in the integer strategy game increases so rapidly with T that it has been conjectured that for $T \geq 10$ it is always possible to find a mixed strategy satisfying the requirements of the Matheson strategy. This conjecture seems unlikely; but in any case, when T is large there should always exist a mixed strategy which comes very close. Even if an exact solution exists, it may not be easy to find such a mixed strategy by trial-and-error methods. There exist possible systematic methods for doing this; for instance, linear programming can be used. However, it does not appear that any attempt has been made to assess the computational feasibility of such methods.

If D (or A) and T are not too large, it is possible to find the optimum offensive and defensive strategies for the integer strategy game by means of the well-known techniques of linear programming — that is, by maximizing a linear function of several unknowns which are also subject to a finite number of linear constraints. This, of course, does not yield a general formula for the strategies in terms of the unknowns T, D, A, q_0 and q_1; nevertheless, in view of the extreme complexity of the formulas in Section 4.3.1, this is usually not much of a disadvantage. However, in many situations it takes too long to obtain a solution by linear programming even using an electronic computer. It appears that the maximum feasible (D,T) is of the order of (25,10).

4.3.4 Generalization of the Preallocation Problem

The Matheson game can be treated as a constrained game, and can be solved easily by linear programming, using a result of Charnes (1953). Linear programming can be applied straightforwardly to solve various generalizations which Matheson did not consider, some of which are given below:

(1) It may be necessary to limit the maximum number of defensive missiles or offensive weapons assigned to any target because of radar traffic-handling limitations or interceptor-attacker geometry.

(2) The defensive missile and offensive weapon assignment doctrines need not be one-on-one. For example, if doctrine i is to be used at a target, one intercepts the first weapon with D_{i1} missiles, the second weapon with

D_{i2} missiles, and so on. The sequence $(D_{i1}, D_{i2}, \ldots)$ defines the ith doctrine. Presumably, $D_{ij} \geq D_{ik}$ for $j < k$; also, $\sum_j D_{ij} = D_i$. The second defensive constraint at the end of Section 4.1 becomes $d = \sum_i D_i x_i$. Other problems that can be dealt with include decoys, shoot-look-shoot, etc.

(3) One can easily handle allocation problems in which there are several different types of defensive missiles (or offensive weapons), each with its own stockpile size and reliability.

(4) Suppose that the defense must protect several different modules, which are located so far apart that the missile stockpile for one module cannot be used for the defense of any other module. Linear programming techniques can be used to find the optimum between-module and within-module allocations. One can also derive a method based on dynamic programming for determining the value of each of the within-module Matheson games (regarded as a function of A_i, the number of offensive weapons allocated that module).

As a more complicated example, consider generalized shoot-look-shoot strategies:

(1) Launch defensive missiles one at a time, observing after each launch whether or not the missile has destroyed the weapon it was directed against. Continue until one of the following events occur:

  (a) the weapon is destroyed,

  (b) the missile supply assigned to that target is exhausted,

  (c) b defensive missiles have been launched.

(2) If the weapon survives b defensive missile launches, launch a final salvo of c defensive missiles (or, the remaining missile supply, if it is less than c).

Let f(i,j) be the probability that a target survives if i weapons and j missiles are allocated to the target and the generalized shoot-look-shoot strategy (b,c) is used. This can be iteratively calculated using the following equation:

$$f(i,j) = \sum_{k=0}^{u-1} (1-\rho)^k f(i-1,j-k-1) + (1-\rho)^u \left(1-(1-\rho)^v p\right) f(i-1,j-u-v) \quad ,$$

where $u = \min(b,j)$ and $v = \max(c,j-u)$. Note that

$$f(0,j) = 1 \qquad \text{for all } j$$

and

$$f(i,0) = (1-p)^i \qquad \text{for all } i \quad .$$

If $b = 1$, $c = 0$, this defensive strategy becomes merely a one-on-one defensive strategy. If $b = c = 1$, the standard shoot-look-shoot defensive strategy results.

This generalized problem can be solved by linear programming very easily. If, however, it is desired to apply Matheson's solution to approximate the solution of this shoot-look-shoot problem, there are at least two ways of doing this:

(1) $E_1(f)$ — assume that the offense allocates weapons to targets according to the Matheson game and the defense, observing this allocation, selects the best possible defensive allocation of missiles to targets using a generalized shoot-look-shoot strategy;

(2) $E_2(f)$ — assume that the defense allocates missiles to targets according to the strategy above, but the offense, observing this allocation, selects the best possible allocation of weapons to targets.

If $E_1(f)$ and $E_2(f)$ are close together, these allocations should be close to the optimum allocations when each side is ignorant of the other's allocation.

4.3.5 The Variation in the Number of Targets Surviving in a Matheson Game

The Matheson game provides the missile defense analyst with the expected number of targets destroyed if both the offense and defense use their optimum mixed strategies to allocate their weapons and missiles. It is frequently of great importance to know the variability that may be expected in an actual engagement — if the expectation is that 35 out of 50 targets will be destroyed, what is the probability that (say) 25 or fewer targets will be destroyed?

In general this is a difficult problem, and analytic results (other than upper bounds) are difficult to obtain. If both the offense and the defense use pure strategies, one can calculate an upper bound to the variance of the number of targets surviving the attack. Specifically, set Z_i equal to 1 if target i survives, and 0 if target i is destroyed; then

$$\mathrm{var}\left(\sum_{i=1}^{T} Z_i\right) \le \frac{T^2}{T-1} V(1-V) \quad ,$$

where V, as before, is the expected fraction of targets saved. Note that if the random variables were independent (which clearly they are not in the Matheson game), then ΣZ_i would be a binomial random variable with variance TV(1-V). It is rather surprising that the actual variance is bounded above by an amount which differs (in percentage terms) very slightly from the binomial value. It is gratifying to note that the variance bound depends only on the expected fraction of targets surviving, not on the actual offensive weapon and defensive missile allocations.

If Z is a positive random variable with $E(Z) = \mu$ and $\mathrm{var}(Z) = \sigma^2$, a form of the Chebychev inequality states that

$$\text{Probability } (Z \ge t) \le \frac{\sigma^2}{\sigma^2 + (\mu - t)^2} \quad \text{for} \quad t \ge \mu \quad .$$

This may be used to set bounds on the probability that the actual number of surviving targets is greater than a preset value t above the mean (or less than a preset value t below the mean):

$$\Pr\left(\sum_{i=1}^{T} Z_i \ge t\right) \le \frac{T^2V(1-V)/(T-1)}{T^2V(1-V)/(T-1) + (t-TV)^2} \quad , \quad t \ge TV \quad ,$$

$$\Pr\left(\sum_{i=1}^{T} Z_i \le t\right) \le \frac{T^2V(1-V)/(T-1)}{T^2V(1-V)/(T-1) + (TV-t)^2} \quad , \quad t \le TV \quad .$$

One should note, however, that Chebychev-type bounds are not very strong.

One can prove that the above arguments are also valid when the offense uses an optimum mixed strategy for allocation and the defense uses an optimum allocation strategy which happens to be pure. Unfortunately, it appears very difficult to prove the corresponding theorem when the optimum allocation strategies for both sides are mixed — a much more realistic case. To get around this difficulty, one can introduce the concept of an approximating pure defense strategy — one which is obtained from the real-valued Matheson allocation $(x_0, x_1, \ldots, x_{m-1})$ by a method which essentially involves rounding up or down to the nearest integer.

First derive an upper bound on the loss (the reduction in the expected fraction of targets saved) incurred by the defense when it uses an approximating pure strategy instead of the optimum mixed strategy determined by the Matheson game. If Δ_i is the difference between x_i and the integer approximating x_i, and V and V* are the respective expected fractions of targets lost using the mixed defense strategy and the approximating pure defense strategy, then the bound is simply

$$|V - V^*| \leq \frac{\sum_{i=0}^{m-1} |\Delta_i|}{2T} = \delta^2 \quad .$$

In most cases, this difference is very small. For example, the optimum defense strategy (9.60, 9.65, 10.55, 11.65, 8.55) associated with the Matheson game for $T = 50$, $A = D = 100$, $q_0 = 0.1$ and $q_1 = 0.955$ leads to the approximating pure strategy (9, 10, 11, 12, 8). For this pair, the difference δ^2 is about 0.001.

Using this bound, one can modify the Chebychev bound (given earlier) by replacing $\sigma^2 = T^2V(1-V)/(T-1)$ with $\sigma^2 = T^2V(1-V)/(T-1) + T^2\left(\delta^2-(V^*-V)^2\right)$.

One can also give a strong plausibility argument that, if both the offense and defense use pure strategies, then, as the number of targets approaches infinity, the probability distribution function of the number saved converges (in a carefully defined sense) to a Gaussian distribution. More specifically,

$$\lim_{T\to\infty} \Pr\left(\frac{Z - VT}{TV(1-V)^{1/2}} \leq t\right) = N(0,\sigma^2) \quad \text{and} \quad \sigma^2 \leq 1 \quad ,$$

where $N(0,\sigma^2)$ denotes the Gaussian distribution with zero mean and variance σ^2. A strong argument can be made (supported by Monte Carlo simulation results) that this statement also applies when the offense uses an optimal mixed strategy and the defense an approximately optimal pure strategy as discussed earlier.

The use of a Gaussian limiting distribution enables one to make more accurate estimates of the probability that the number of surviving targets is less than (or greater than) a preset value t. However, the estimated probability may be either high or low, whereas the corresponding probability calculated from the Chebychev inequality is guaranteed to be an upper bound. To illustrate the difference, consider again the Matheson game with parameters

$T = 50$, $A = D = 100$, $q_0 = 0.1$ and $q_1 = 0.955$, for which $V \approx 0.582$ or about 29 targets. In this game, the Chebychev bound leads to the statement that $Pr(\Sigma Z_i \leq 19) \leq 0.108$, whereas the Gaussian limiting distribution leads to $Pr(\Sigma Z_i \leq 24.5) \approx 0.093$.

4.3.6 Other Models of Preallocation Offense and Defense

Matheson has provided the most comprehensive description of the optimum offensive and defensive strategies required when neither side knows the other's allocation. However, it is worth noting that other analysts have also studied this allocation problem. In the literature, this allocation problem is frequently labeled a "Blotto game" or a "Colonel Blotto game"; apparently this name was introduced by Caliban (a pseudonym of the English puzzle-constructor Hubert Phillips) in his Weekend Problems Book. Blackett (1954) solves a simple Blotto game in a logistics context. Translated into a missile allocation problem, the defense constructs one real target and (T-1) dummy targets, and the offense, not knowing which target is real, distributes A weapons among these T targets.

Blotto games can be formulated in two different ways. Matheson considered a discrete Blotto game in which the offense and defense levels allocated to the various targets were restricted to integer values. However, one can consider continuous Blotto games in which this requirement is relaxed: at each target, weapons and missiles are represented by real numbers. Continuous Blotto games are analytically more tractable but rather more difficult to interpret. For example, how much damage is done to a target attacked by 3.72 weapons and defended by 1.17 missiles? Usually, q_0 is set equal to zero and q_1 is set equal to unity; the target is assumed to be saved if the number of missiles allocated to it is greater than or equal to the number of weapons, and destroyed if the number of missiles allocated to it is less than the number of weapons. According to Luce and Raiffa (1957), Borel posed a problem of this type for three targets in 1921.

Galiano (1967b) and Penn (1971) independently derive Matheson's optimum offense and defense allocations for the discrete Blotto game when $q_0 = 0$ and $q_1 = 1$. In addition, they derive the corresponding allocations and the expected fraction of targets saved in the continuous Blotto game when $q_0 = 0$ and $q_1 = 1$. If the offense is dominant ($a \geq d$), then he attacks a typical target with a_i weapons, where a_i is a real number selected according to the uniform probability density function between 0 and 2a. The defense defends a typical target with probability d/a; if defended, a target is allocated d_i missiles, where d_i is a real number selected according to the uniform probability density function between 0 and 2a. The expected fraction of targets saved is $E(f) = d/2a$. If the defense is dominant ($a \leq d$), then he defends a typical target with d_i missiles, where d_i is a real number selected according to the uniform probability density function between 0 and 2d. The offense attacks a typical target with probability a/d; if attacked, a target is allocated a_i weapons, where a_i is a real number selected according to the uniform probability density function between 0 and 2d. The expected fraction of targets saved is $1 - a/2d$.

Schreiber (1968) calculates some additional quantities of interest for the continuous Blotto game when $q_0 = 0$ and $q_1 = 1$. The expected number of unused defensive missiles per target defended is equal to $d/3$ (when $a \geq d$) and $d - 2a/3$ (when $a < d$). The expectec number of unintercepted weapons per target is $d - 2a/3$ (when $a \geq d$) and $a/3$ (when $a < d$).

Galiano (1969) suggests that the continuous Blotto game can also be used when the probability of target destruction by an unintercepted weapon is not unity but p. The expected target damage can be approximated by increasing the number of defensive missiles per target from d to $d - 1/\log_e(1-p)$, and using the results cited above.

Note that the continuous Blotto game is biased in favor of the offense; the offense can destroy a target with an infinitesimal excess of weapons over missiles. Penn (1967b) points out the discrete and continuous Blotto games can be made comparable to each other if, in the discrete game, one-half of the target is awarded to the offense when the number of weapons is equal to the number of missiles. Curiously, the optimum strategies that Penn derives shun ties — the offense always allocates an odd number of weapons and the defense frequently allocates an even number of missiles.

Schreiber (1968) derives the general form of the optimum offense and defense strategies for the continuous Blotto game with one-on-one engagements, assuming a broad class of damage functions. Specifically, he assumes that $P(x,y)$, the probability that a target survives when attacked by y weapons and defended by x missiles, is of the form

$$P(x,y) = s(y) \qquad 0 \leq y \leq x \quad ,$$

$$P(x,y) = s(x)t(y-x) \qquad x \leq y \quad .$$

The functions $s(x)$ and $t(y)$ are assumed to be convex with continuous derivatives, and $s(0) = t(0) = 1$. Therefore, $P(x,y)$ is continuous in x and y, but $dP(x,y)/dy$ is discontinuous at $x = y$. At the beginning of this chapter, $s(y)$ was assumed to be q_0^y and $t(y-x)$ was assumed to be q_1^{y-x}.

Let $f(y)dy$ be the fraction of targets attacked by y weapons, and $g(x)dx$ the fraction of targets defended by x missiles. These functions define offensive and defensive strategies (analogous to the x_i and y_i used by Matheson); they are in general discontinuous and contain delta-functions. Schreiber shows that that the optimum defense strategy is given by $g(x)$ satisfying the equation

$$G(y) = \int_0^\infty g(x)P(x,y)dx = m - hy \quad ,$$

in some interval $U < y < V$ subject to the constraints $G(y) \geq m - hy$ outside $U < y < V$,

$$\int_0^\infty g(x)dx = 1 \quad \text{and} \quad \int_0^\infty xg(x)dx = d \quad .$$

The quantities m, h, U and V are determined so that $m - ha$ is as large as possible. The corresponding optimum offense strategy is given by $f(y)$ satisfying the equation

$$F(x) = \int_0^\infty f(y)P(x,y)dy = n + kx \quad ,$$

in some interval $U' < x < V'$ subject to the constraints $F(x) \leq n + kx$ outside $U' < x < V'$,

$$\int_0^\infty f(y)dy = 1 \quad \text{and} \quad \int_0^\infty yf(y)dy = a \quad .$$

The quantities n, k, U' and V' are determined so that $n + kd$ is as small as possible.

Schreiber demonstrates that $U = U'$ and $V = V'$. In addition, he shows that

$$g(x) = \lambda\delta(x-0) + (1-\lambda)\phi(x) \quad ,$$

where $\delta(x-x_0)$ is the unit delta-function at $x = x_0$, $0 \leq \lambda < 1$, and $\phi(x) > 0$ for $U < x < V$, $\phi(x) = 0$ elsewhere, contains no delta-functions and integrates to unity. Finally, he shows that

$$f(y) = \mu\delta(y-U) + \nu\delta(y-V) + (1-\mu-\nu)\theta(y) \quad ,$$

where $0 < \mu < 1$, $0 < \nu < 1$, and $\theta(y) > 0$ for $U < y < V$, $\theta(y) = 0$ elsewhere and $\theta(y)$ contains no delta-functions and integrates to unity. Note that the optimum defense strategy can be realized no matter what the value of d, but the optimum offense strategy is impossible to realize unless $a > U$. When $a \leq U$, the optimum offense strategy depends on $s(y)$ alone, and must be derived by a special argument.

The forms of the optimum offense and defense strategies and the value of U can be specified in slightly greater detail by noting in which of four disjoint regions (d,a) is located:

1)	$a > y_c$, $d < U$	(super stable defense)
2)	$a > y_c$, $U < d < x_c$	(strong stable defense)
3)	$a < y_c$, $d < x_c$	(weak stable defense)
4)	$d > x_c$	(unstable defense)

where x_c and y_c are critical values which can be calculated from the optimum strategies.

4.4 SOME NONPREALLOCATION STRATEGIES

When neither side knows the other's allocation, it is clear from the preceding section that the optimum offensive and defensive preallocation strategies are not easy to derive. This is not a particularly serious objection to their use, since they can be calculated well in advance of the actual attack. However, another difficulty can occur. It may not be possible to determine the aiming-point of a weapon accurately enough to decide what its target is. Preallocation defenses require that the defense keep track of exactly how many offensive weapons have been directed at each target and were engaged in order to decide whether or not to allocate a defensive missile against the next weapon approaching that target. Is it possible that there exist strategies which are nearly as good as (or better than) the Matheson preallocation strategy but are much easier for the defense to implement?

4.4.1 A Group Preferential Defense Strategy Against an Offense That Can Vary His Attack Size

One possible defense strategy is the following group preferential strategy. Allocate the entire stockpile to the defense of a fraction d/k of the targets where k is an integer greater than or equal to d; when any target in this subset is attacked, defend it with a missile as long as any remain in the stockpile. The subset of targets to be defended is selected at random; however, there is clearly a great advantage to selecting a geographically compact subset in order to minimize the number of decisions that must be made concerning the target against which each offensive weapon is directed. It is assumed that the offense knows the choice of k, but not the actual random subset being defended. The word "preferential" has been applied to a great many different types of strategy. Indeed, most of the strategies of this chapter would fall under somebody's definition of preferential strategies. For this reason, the word has been avoided in this work, except in this section. It should be noted, however, that preallocation strategies are perhaps those the most frequently designated as preferential.

Unfortunately, it is quite difficult to make a direct comparison between this strategy and the Matheson preallocation strategy in terms of the original model. The difficulty arises because it is now necessary to consider the order in which the offensive weapons arrive. To get around this analytic difficulty, one can replace the actual attack with a hypothetical attack of somewhat greater severity and analyze the latter problem instead. In addition, only integral k will be permitted; this can be a severe restriction.

To be specific, assume that the offense attacks the group of targets in "waves," each consisting of a single weapon against each target. Also assume that the offense attacks the targets with a total of i different waves. The quantity i is a random variable with a probability density function which can be selected by the offense but which must have a mean of a: $\Sigma i y_i = a$. In other words, the offense attacks the entire set of targets with the same type of strategy that he earlier used on individual targets in the set. Obviously, this strategy is realistic only if the enemy has a large number of equal groups of targets to attack, not just one.

It is not too difficult to prove that the optimum offensive strategy against this simple defense is not a multilevel strategy such as Matheson's, but instead a strategy containing at most a lower and an upper attack level, which are denoted by i and (a+j). This is a consequence of the fact that the offense knows the defensive parameter k. It is also not difficult to see that if the defense uses a preallocation strategy against this hypothetical attack, the analysis of Matheson goes through exactly as before (in the limit when the number of groups of targets is large). Therefore, a comparison between the preallocation defense strategy and the group preferential defense strategy proposed above remains valid.

The expected fraction of targets saved can be expressed in the following terms:

$$E(f) = \max_{k} \min_{i,j} \left(\frac{j}{a-i+j} \left(\left(1 - \frac{d}{k}\right) q_0^{i} + \frac{d}{k} q_1^{\min(i,k)} q_0^{\max(0,i-k)} \right) + \frac{a-i}{a-i+j} \left(\left(1 - \frac{d}{k}\right) q_0^{a+j} + \frac{d}{k} q_1^{\min(a+j,k)} q_0^{\max(0,a+j-k)} \right) \right) ,$$

where k, i and j are all integers, $k \geq d$ and $0 \leq i \leq a$. (When j is equal to zero and i is equal to a, the above formula must be modified slightly to avoid terms of the form 0/0.)

To illustrate the use of the above formula, consider a simple example: let $d = 1$, $a = 3$, $q_1 = 1$ and $q_0 = 2/3$. First consider the outcome of using techniques discussed in previous sections. From Section 4.2.0, it is easy to see that the expected fraction of targets saved when the offense has the last move is $4/9 = 0.444$, and the expected fraction of targets saved when the defense has the last move is $(1/3) + (2/3)(2/3)^3 = 0.531$; these represent upper and lower bounds to the problem. If neither side knows the other's allocation and preallocation strategies are used, so that Matheson's solution applies, then the offense strategy is $y_1 = y_2 = y_3 = 1/6$, $y_4 = 1/2$, the defense strategy is $x_0 = 1/2$, $x_1 = x_2 = x_3 = 1/6$, and the expected fraction of targets saved is $1/2 = 0.500$. Turning now to the group preferential strategy being considered, one can determine by trial and error that the expected fraction of targets saved is equal to $239/486 = 0.491$, which corresponds to a choice of k equal to 3 (that is, one-third of the targets defended) and $i = 2$, $j = 1$, giving offensive levels of 4 and 2 each with a probability of 1/2. In this example, the group preferential strategy is only slightly inferior to the Matheson strategy, and it may well be much easier to implement. In other examples, the group preferential defense might be superior to the Matheson strategy.

Intuitively it seems logical to set $k = a$ if a is integral, because the defensive stockpile will then be equal to the expected attack size on the defended subset. However, as d approaches a, the advantages of randomization are lost; when the two are equal, the offense knows the defense strategy. By allowing k to vary, an additional degree of freedom is given the defense. Of course, if $d \geq a$, the best defense is to engage each weapon, abandoning the group preferential defense.

What if the offense does not know the level k that the defense has selected? In such a situation, a possible response of the offense is to tailor his attack so that the expected fraction of targets saved is the same no matter which level, k, the defense selects (over a range of values, say $d \leq k \leq t$). The offense can easily achieve this goal by selecting the Matheson preallocation strategy corresponding to $d = h$, $t = m$. It can be shown that when $q_1 = 1$, the expected fraction of targets saved when the defense selects a value of k such that $d \leq k \leq t$ and the offense uses a Matheson strategy is:

$$E(f) = \frac{d}{t\left(1-q_0^d\right) + dq_0^d} .$$

The value of t is determined by the size of the offensive stockpile:

$$a = \sum_{i=d}^{t} i y_i \quad .$$

To illustrate this strategy, consider the following simple example: let $d = 2$, $t = 4$, $q_1 = 1$ and $q_0 = 1/2$. The strategy used by the offense is $y_2 = 7/10$, $y_3 = 2/10$ and $y_4 = 1/10$, corresponding to an overall offensive level, a, of 12/5. Since the attack level is variable, the above strategy can only be implemented, even approximately, when there are several independent groups of targets. The expected fraction of targets saved is equal to 4/7 = 0.573. However, if one uses optimal (Matheson) preallocation defense and offense strategies corresponding to $d = 2$, $a = 12/5$, $q_1 = 1$, and $q_0 = 1/2$, the offense and defense strategies are

$$y_0 = 7/25, \quad y_1 = y_2 = y_3 = y_4 = 3/25, \quad y_5 = 6/25 \quad ,$$

$$x_0 = 3/10, \quad x_1 = x_2 = x_3 = x_4 = 3/20, \quad x_5 = 1/10 \quad ,$$

and the expected fraction of targets saved is increased to 0.640. In other words, the defense has paid a price in effectiveness of 0.067 by using a group preferential strategy instead of a Matheson strategy. In another example, the group preferential strategy might actually be superior. Note that the defense should randomize its choice of k in the range $d \leq d \leq t$ in order to prevent the offense from taking advantage of a constant value of k.

4.4.2 A Group Preferential Defense Strategy Against an Offense of Fixed Size

Can any analysis be made of defensive area strategies when the offense model of the beginning of this section holds — that is, when the offense attacks a single group of targets with a stockpile of fixed size? As mentioned earlier, the outcome depends upon the order of arrival of offensive weapons. Consider two extreme cases: (1) offensive weapons arrive at random; (2) the offense has complete control over the order of arrival. Assume that each side knows the other's stockpile size but not the specific allocation of weapons to targets, or the specific subset of targets selected for area defense. Assume also that $q_1 = 1$ and $q_0 = 0$ to simplify the calculations.

The following numerical example may be instructive. Assume that one has four targets which can be defended by five missiles, and assume that the offense has ten weapons. Matheson (1966) has worked out by combinatorial arguments the optimum offense and defense strategies and the expected fraction of targets saved if both sides use preallocation strategies. The offense randomly assigns (4,3,2,1) weapons; the defense randomly assigns (3,2,0,0) missiles with probability 2/3, (3,1,1,0) missiles with probability 1/6, and (1,2,1,1) missiles with probability 1/6; the expected fraction of targets saved is 5/16, or 0.3125. This result will be compared with the outcomes of offense models (1) and (2) above.

If the offense has complete control over the order of weapon arrival, the best thing for him to do is to arrange matters so that the last four weapons are allocated one each to the four targets. If at least five or his first six weapons all happen to be directed at targets in the defended subset, then the offense will destroy the entire set of targets. Since he does not know which targets are being defended, he will not ordinarily be this fortunate. In this example, all possibilities can be enumerated with relative ease.

For example, assume the offense attacks the four targets with (4,2,2,2) weapons. There are six equiprobable ways in which the defense can select two targets to be defended. If a (4,2) pair is selected, the offense causes the defense to use four defensive missiles against the first six offensive weapons, and the fifth defensive missile can be used to save one target. If a (2,2) pair is selected, the offense causes the defense to use only two defensive missiles against the first six offensive weapons, and the remaining three defensive missiles can be used to save both targets (and have one defensive missile left over). Note that the two undefended targets are always destroyed by the offense. The expected fraction of surviving targets is, then, (1/2)(1/4) + (1/2)(1/2) = 3/8. The various possibilities are summarized in the table below.

EXPECTED FRACTION OF TARGETS SAVED

(Controlled Offensive Arrival Order)

Offense Strategy	Number of Targets Defended 1	2	3
(3,3,3,2)	0.250	0.458	0.125
(3,3,3,1)	0.250	0.375	0.188
(4,3,2,1)	0.250	0.375	0.188
(4,2,2,2)	0.250	0.375	0.125
(4,4,1,1)	0.250	0.417	0.250
(5,2,2,1)	0.250	0.292	0.188
(5,3,1,1)	0.250	0.333	0.250
(6,2,1,1)	0.188	0.250	0.188
(7,1,1,1)	0.188	0.250	0.188

The best strategy for the defense is to defend two targets, so the best strategy for the offense is either (6,2,1,1) or (7,1,1,1) — he can insure that the expected fraction of surviving targets is at most

0.250 no matter what size subset the defense selects. (If the defense selects either one or three targets to defend, the offense can actually do better.) In other words, if the offense can control the arrival order of this weapon the defense loses 0.062 by shifting from a preallocation to a group preferential strategy.

Note, however, that the optimum offense strategy is extremely unbalanced. If the defense were to shift to a preallocation defense using (2,2,2,2), it would be a disaster for the offense — at least three-quarters of the targets would be saved every time. Therefore, the offense dares not use the strategy (7,1,1,1) or (6,2,1,1) unless he is certain that the defense is using a group preferential strategy. If the offense conservatively uses (4,3,2,1), then the best group preferential strategy yields an expected fraction of targets saved of 0.375 — considerably better than a preallocation strategy. In other words, the decision as to whether to use a group preferential or a preallocation defense strategy depends upon what the defense thinks the offense knows about its plans — a notably difficult matter to assess.

What if the offense weapon arrivals are random? Assume that the offense attacks with (4,3,2,1) weapons. There are, as before, six equiprobable ways in which the defense can select two targets. If (1,2), (1,3), (1,4) or (2,3) are defended, then both targets are saved. If (2,4) is defended then one target is saved. If (3,4) is defended, three possibilities occur. The last two weapons to arrive can be split between the two targets, in which case neither target survives; the last two weapons to arrive can both be directed at the four-weapon target, in which case the three-weapon target survives; the last two weapons to arrive can both be directed at the three-weapon target, in which case the four-weapon target survives. These three events happen with probabilities $(3/7)(4/6) + (4/7)(3/6) = 4/7$, $(4/7)(3/6) = 2/7$, and $(3/7)(2/6) = 1/7$, respectively, if the arrival order is random; the expected fraction of surviving targets if (3,4) is defended is equal to $(0)(4/7) + (1/4)(2/7) + (1/4)(1/7) = 3/28$. The expected fraction of surviving targets is, then, $(2/3)(1/2) + (1/6)(1/4) + (1/6)(3/28) = 11/28$. The various possibilities are summarized in the table below. The first column is, of course, the same as in the earlier table. The best pure strategy for the offense is (5,2,2,1) — he can insure that the expected fraction of surviving targets is at most 0.348 no matter what size subset the defense selects. In other words, the defense gains 0.036 by shifting from a preallocation to a group preferential strategy if he can be certain that the offense cannot control the arrival order of its weapons. By using a mixed strategy, the offense could hold the defense to somewhat less, but no less than 0.335, which is still better for the defense than a preallocation strategy.

Again, the offense may be conservative and use the strategy (4,3,2,1) to protect against a knowledgeable defense. If he does, then the defense can insure that the expected fraction of targets saved is 0.392.

EXPECTED FRACTION OF TARGETS SAVED

(Random Offensive Arrival Order)

Offense Strategy	Number of Targets Defended 1	2	3
(3,3,2,2)	0.250	0.458	0.244
(3,3,3,1)	0.250	0.375	0.285
(4,3,2,1)	0.250	0.392	0.308
(4,2,2,2)	0.250	0.375	0.273
(4,4,1,1)	0.250	0.422	0.329
(5,2,2,1)	0.250	0.335	0.348
(5,3,1,1)	0.250	0.342	0.365
(6,2,1,1)	0.188	0.325	0.370
(7,1,1,1)	0.188	0.330	0.395

To sum up this example, it is likely to be profitable for the defense to shift from a preallocation to a group preferential strategy — the defense loses only against a well-informed offense which can exercise a high degree of control over its attack order.

4.4.3 A More General Class of Nonpreallocation Strategies

There is no need for the defense to restrict himself to a nonpreallocation strategy in which the entire missile stockpile is used to defend a subset of the targets. In fact, the defense can select a mixture of preallocation and nonpreallocation strategies in the following manner. Divide the target set into m disjoint groups of size $t_1, t_2, \ldots, t_m$, and allocate d_i weapons to the defense of the ith group. Note that

$$\sum_{i=1}^{m} t_i = T \quad \text{and} \quad \sum_{i=1}^{m} d_i = D \quad .$$

Here T, D, and A represent the (unnormalized) numbers of targets, defensive missiles, and offensive weapons. If all $t_i = 1$, then one is dealing with a preallocation strategy; if $m = 2$ and one of the $d_i = D$, then one is dealing with the group preferential strategy discussed in Section 4.4.2. A more general strategy is specified by (t_i, d_i), $i = 1, \ldots, m$: the assignment of targets to the m disjoint groups is done at random.

The offense is assumed to use a preallocation strategy: a fraction y_i of the targets is attacked by i weapons, and the assignment of targets to weapons is done at random. Thus, an offense strategy is specified by $(y_0, y_1, \ldots)$, where $\Sigma i y_i = A/T$. For simplicity, assume that the attack takes place in two waves, the second wave consisting of exactly one weapon on each target for which $i > 0$ (offensive arrival order controlled).

It appears quite difficult to determine the optimum offensive and defensive strategies as a function of A, D and T if $D < A$ and $T < A$. For defense-last-move, Analytic Services Corporation (1968) has shown that the determination of an optimum offense strategy is equivalent to solving a set of equations which is nonlinear in the y_i; the computational problems become formidable for all but the simplest examples.

The following example is given in Analytic Services Corporation (1968). Assume that $q_0 = 0$ and $q_1 = 1$, so the defense is one-on-one. Let $D = 4$, and let $A = 2T$; what strategy $(y_0, y_1, \ldots)$ should the offense select if the defense has the last move? Assume that at least one and at most four weapons are assigned to each target, and that the defense, knowing the strategy $(0, y_1, y_2, y_3, y_4, 0, \ldots)$ but not which targets will actually be attacked by i weapons, selects one of the following three strategies:

(1,1), (1,1), (1,1), (1,1), (T-4,0)	(preallocation)
(1,2), (1,2), (T-2,0)	(preallocation)
(2,4), (T-2,0)	(nonpreallocation)

Assume that the offense wishes to insure that the expected number of targets saved per defense missile used is the same for all three strategy components (1,1), (1,2) and (2,4). Equating the first and second components in terms of the expected number of targets saved, one has

$$2y_1 = y_1 + y_2 \quad .$$

Equating the first and third components, one has

$$4y_1 = 2\left(y_1^2 + 2y_1y_2 + 2y_1y_3 + y_2^2\right) + \left(2y_1y_4 + 2y_2y_3\right) \quad .$$

The first term in parentheses is the probability that the two targets with the (2,4) component defense will be attacked by a total of two, three or four weapons; the second term is the probability that they will be attacked by five weapons (and hence only one target saved). The right-hand side of this equation is strictly correct only in the limit as $T \to \infty$. The attack restrictions on the y_i are:

$$y_1 + y_2 + y_3 + y_4 = 1$$

$$y_1 + 2y_2 + 3y_3 + 4y_4 = 2 \quad .$$

Solving these four equations in four unknowns, one finds the optimum offense strategy (0, 3/8, 3/8, 1/8, 1/8, 0,...). The expected number of targets saved is 3/2. Note that this strategy can be implemented exactly only when T is an integral multiple of 8.

It would be desirable to solve this problem without the restrictions on the minimum and maximum attack size and the allowable defense strategies. It seems likely that the optimum offense strategy and the expected number of targets saved would be unchanged.

However, it is clear that if one considers the corresponding offense-last-move problem, the above limitations on the offense strategies are unnecessarily restrictive. It is conjectured that if the defense is allowed to choose any strategy (t_i, d_i), $i = 1, 2, \ldots, m$, and if the offense knows this strategy but not the specific target assignments, then the optimum defense strategy is (2,4), (T-2,0) and the optimum offense response is (0, 3/4, 0, 0, 0, 1/4, 0,...). The expected number of targets saved ranges from 1 (if $T = 4$) to 9/8 (as $T \to \infty$).

If neither side knows the other's strategy before choosing its own, then one is dealing with a game-theoretic problem. The expected number of targets saved will lie between the offense-last-move and defense-last-move values, and both sides must use mixtures of strategies. In general, these game-theoretic problems are more difficult to solve. In Analytic Services Corporation (1968), eleven equations in eleven unknowns are solved to determine the optimum mix of the three allowable defense strategies as well as the optimum mix of two offense strategies of the form $(0, y_1, y_2, 0, y_4)$ and $(0, y'_1, y'_2, y'_3, 0)$. The expected number of targets saved turned out to be 1.4883 — very close to the defense-last-move value.

In a later Analytic Services Corporation report, Israel (1970) describes a linear program to determine approximately optimum nonpreallocation defense strategies when targets are resistant to damage ($q_0 > 0$) and missiles are unreliable ($q_1 < 1$). As mentioned earlier, the expected fraction of targets surviving is not linear in the offensive allocations y_i; it is linear in the sense that a mixed offense strategy is a linear combination of pure offense strategies. Unfortunately, this means that an exact linear programming solution to the offense and defense allocation problem must consider as many linear constraints as there are pure offense strategies — a truly astronomical number. The linear program devised by Israel is much more modest — it only considers a relatively few pure offense strategies, selected to have intuitively desirable properties. It is difficult to determine how close to the optimum one comes except for extremely small values of A, D, and T. He uses this program to infer plausible rules of thumb such as the following:

Typically, allocations (t_i, d_i) satisfy the equation $d_i/t_i = \max(A/T, D/T)$. In one example ($T = 24$, $A = 108$, $0 \le D \le 108$, $q_0 = 0$, $q_1 = 1$), the expected fraction of targets saved using an approximately optimum nonpreallocation strategy lies about halfway between a defense-last-move strategy and the Matheson game strategy.

4.4.4 A Nonpreallocation Strategy Involving a Stockpile of Defensive Missiles Held in Reserve

Penn (1969) has also considered a mixture of nonpreallocation and preallocation defense strategies. In order to simplify the analysis, he assumes a continuous Blotto game (that is, offensive weapons and defensive missiles can be assigned in fractions, not just integers); furthermore, he assumes perfect interceptors and soft targets ($q_0 = 0$, $q_1 = 1$). The offense has a weapons available per target, and the defense has d missiles per target which must be preallocated, plus a reserve of r missiles per target which need not be preallocated to any specific target. As before, one must distinguish between attack dominance, when some targets must be left undefended, and defense dominance, when all targets are defended. Penn assumes that the offense arrival order is controlled rather than random: specifically, he assumes the offense saves back ϵ weapons per target until the end of the attack, and then applies these uniformly over the targets.

Although this model has been made as simple as possible, Penn is unable to derive the optimum offense and defense strategies except when these strategies have been substantially restricted. However, it is interesting to note that the expected fraction of targets saved is the <u>same</u> for two rather different restricted classes of defense strategies, suggesting that the result has more general applicability. These two different restricted classes of defense strategies are:

(1) <u>Tapered Defense Strategy Class</u>: Allocate the d preallocation missiles per target as if no reserves were available (see Section 4.3.6, in particular the reference Penn (1971)). Each target is also promised reserve missiles up to a specified maximum amount. If <u>all</u> targets actually received their maximum reserve assignments, a total of $r' > r$ reserve missiles per target would be needed (in short, more are promised than can be delivered). The maximum number of reserves promised to each target is calculated by pretending that one has a defensive stockpile of $d + r'$ missiles per target, and preallocating all these missiles (as in Section 4.3.6); the difference between the d-missile preallocation and the $(d+r')$-missile preallocation at a target determines the number of reserve missiles promised to that target.

Once the missiles preallocated to a target are used up, they are assigned reserve missiles as needed up to the number promised, provided any missiles in the reserve stockpile remain.

(2) Defend-to-the-Death Strategy Class: Allocate the d preallocation missiles per target to a fraction x of the targets (see Section 4.4.1); allocate the r reserve missiles per target to the remainder of the targets, with the understanding that any reserve missile defends any target (no matter how heavily attacked) as long as any reserves remain. The offense, of course, does not know which targets have been selected in the fraction x.

First, Penn guesses the optimum defense strategy for each class, and proves that using these two strategies with offense-last-move the expected fraction of targets saved is

$$E(f) = \frac{d}{2a} + \frac{r}{a} \qquad \text{if} \quad d + r \leq a \quad \text{(offense dominant)} \quad ,$$

$$E(f) = 1 - \frac{(a-r)^2}{2ad} \qquad \text{if} \quad d + r \geq a \quad \text{(defense dominant)} \quad .$$

If the defense is dominant, the above formula assumes $r \leq a$; if $r > a$, $E(f) = 0$. In tapered defense, about twice as many reserves are promised as are actually available: $2ar/(a-r) + r^2d/(a-r)^2$ for offense dominance, $2r + r^2/d$ for defense dominance. In defend-to-the-death, on the other hand, one allocates the reserve missiles to a fraction r/a of the targets, so that exactly enough reserves are available to counter the attack. Unless the attack is uniform, this will be true only in the limit as the number of targets approaches infinity (and variations in the size of the attack on this fraction become insignificant).

The above calculations assume that d and r have been fixed in advance. This is reasonable, for instance, if d represents short-range missiles that can defend any target in the group. (Models of this type are discussed in more detail in Sections 6.3.1-6.3.5.) If, however, the defense has a free hand to set the levels d and r, subject to the constraint $d + r = c$, then he always should choose $d = 0$, $r = c$. In this case, $E(f) = c/a$ if the offense is dominant and $E(f) = 1$ if the defense is dominant.

Finally, Penn guesses the optimum offense strategy with defense-last-move, obtaining the same E(f) as above.

4.5 DEFENSE DAMAGE ASSESSMENT STRATEGIES

In the preceding sections, it was assumed that the defense has no way of evaluating target damage during the course of an engagement: as a consequence, defensive missiles may be wasted on a target which has already been destroyed. It is obvious that the defense should be able to increase the expected fraction of targets saved if he defends only undestroyed targets. Is the gain in the expected fraction of targets saved large enough to make the cost of this additional real-time data processing worthwhile?

4.5.1 Damage Assessment Strategies When the Attack is Known to the Defense

Krumm (1969) has developed a rather general defense-last-move damage assessment model. In particular, he assumes that both the defense and the offense know the values of A, D, T, q_0 and q_1. Further, he assumes that the defense knows that the offense will attack in a waves, allocating one offensive weapon to each target on each wave. The defense can assess damage after each wave, but the offense has no way of knowing which targets he has destroyed. (Actually, Krumm's model is slightly more general — he assumes that the defensive missile reliability, q_1, monotonically decreases with each successive wave of the attack, and that both of the offense and defense know these changing probabilities. On the other hand, Abramson and Shapiro (no date) consider the same damage assessment model as Krumm, but restrict themselves to perfect defensive missiles — q_1 is always equal to unity.)

The value of E(f) calculated by the following formulas should be viewed with some caution, particularly for small values of A, D and T. E(f) is derived under the assumption that the number of targets surviving after each wave is exactly given by the expected value. Note that this model is not a probabilistic one but a deterministic one. In reality, the number of survivors will differ somewhat from the expected value, and it is possible that the defensive missiles will run out before the defense of the final wave is completed. Alternatively, the defense may end up with unexpended missiles at the end of the engagement. A calculation of E(f) taking the actual engagement variability into account appears to be prohibitively difficult.

It is intuitively reasonable (and Krumm, in fact, proves) that the optimum defense has the following general form when $q_0 > 0$. For analytical convenience, the waves are numbered backwards — wave a arrives first and wave 1 arrives last in the attack.

Waves a through $n+1$: defend no targets
Wave n: defend a fraction of the surviving targets
Waves $n-1$ through 1: defend all surviving targets

Notice that when $q_0 = 0$, the defense must start on the first wave of the attack; otherwise, there will be nothing left to defend at stage two. In this case, the strategy consists of defending a certain fraction of targets starting at wave a. The offense is assumed to be ignorant of which targets are actually selected for defense.

To determine the optimum defense strategy, one must first know the value of n — when should one stop observing and start defending? Calculate the sequence

$$Q_i = \frac{1 - \left(q_0 + (1-q_0)q_1\right)^i}{(1-q_0)(1-q_1)} q_1^{a-i}$$

for $i = a, a-1, \ldots, 1$, and set n equal to the smallest value of i for which $Q_i \geq d$.

Let d_i^*, $i = n, n-1, \ldots, 1$ denote the expected number of defensive missiles to allocate to targets on the ith wave. Krumm shows that $d_n^* + d_{n-1}^* + \ldots + d_1^*$ equals the defensive stockpile, D, and gives a set of recursive equations to determine the d_i^* by:

$$d_n^* = T \frac{d(1-q_0)(1-q_1) - \left(1 - \left(q_0+(1-q_0)q_1\right)^{n-1}\right)q_1^{a-n+1}}{(1-q_0)(1-q_1) + \left(1 - \left(q_0+(1-q_0)q_1\right)^{n-1}\right)(1-q_0)q_1} ,$$

$$d_{n-1}^* = Tq_0^{a-n+1} + (1-q_0)q_1 d_n^* ,$$

$$d_{n-i}^* = d_{n-i+1}^*\left(q_0 + (1-q_0)q_1\right) \quad \text{for} \quad i = 2,3,\ldots,n-1 .$$

Finally, Krumm derives the expected fraction of targets surviving if this strategy is used:

$$E(f) = q_0^a + \frac{1}{T}(1-q_0)q_1 \sum_{j=1}^{n} q_0^{j-1} d_j^* .$$

Alternatively, one can write

$$E(f) = \frac{1}{T} d_1^*\left(q_0 + (1-q_0)q_1\right) .$$

Krumm makes the related point that the defensive damage assessment strategy could be improved if it were revised at the end of each attacking wave, taking as input the number of surviving targets and number of remaining defensive missiles.

It is of interest to calculate the maximum value of D required by the defensive damage assessment strategy — that is, how many defensive missiles are needed if all targets are defended on all waves?

$$d_{max} = \frac{D_{max}}{T} = \frac{1 - \left(q_0 + (1-q_0)q_1\right)^a}{(1-q_0)(1-q_1)} \quad .$$

As expected, D_{max} is less than or equal to A since destroyed targets are attacked but not defended in later waves.

Under what conditions should the defense launch more than one defensive missile against an incoming weapon? Krumm points out that there may exist conditions under which it is more advantageous (in terms of increasing E(f)) to defend some targets during the final wave attack with two missiles than to defend targets during one of the previously undefended waves. In particular, he shows that:

(1) if n = 1, single launches should be used at wave 1 and excess defensive missiles assigned to wave 2 under all circumstances:

(2) if n = 2, single launches should be used at wave 1 and excess defensive missiles assigned to wave 3 if $q_1^2 \geq 1 - q_1/(1-q_0)$.

The expected fraction of targets saved using a preallocation strategy can be compared with the (larger) expected fraction of targets saved using a damage assessment strategy. Numerical calculations show that the difference in expected values never exceeds 0.086 for any value of a, d and q_0, if $q_1 = 1$. In short, damage assessment strategies gain little for the defense in the defense-last-move model. It is likely that a more extensive table including $q_1 < 1$ would tell much the same story.

4.5.2 Damage Assessment Strategies Against Attacks of Unknown Size

An important drawback of Krumm's model is his assumption that the defense knows the exact offense allocation before the start of the attack. The models discussed in this section modify this assumption somewhat. In particular, assume that both the offense and the defense know the values of a, d, q_0 and q_1, but that the defense no longer knows the allocation of offensive weapons to targets before the start of the engagement. To make the problem analytically tractable, it is again necessary to postulate the hypothetical attack introduced in Section 4.4.1: instead of allocating different numbers of weapons to the various targets, the offense attacks the group of targets in "waves," each consisting of a single weapon against each target. Furthermore, the number of waves in the attack is a random variable, i, having a probability density function that can be selected by the offense but which must have a mean value of a: $\Sigma i y_i = a$. In other words, the offense attacks the entire set of targets with the same type of strategy that he earlier used on individual targets in the set.

The damage assessment strategy of Section 4.5.1, it should be noted, requires that the defense know the offense level against the targets in the group. One can determine the effect of an incorrect assumption by the defense about the offense level, and consider how the offense might take advantage of such an incorrect assumption. Modifying the formulas introduced in the defense-last-move model, it is easy for both sides to calculate the expected fraction of targets saved if the defense assumes an attack level n and the offense actually uses an attack level i. Let s denote the first defended level corresponding to an assumed attack level n; then

$$E_i(f) = q_0^i \quad \text{for} \quad i = 1,2,\ldots,s-1 \quad ,$$

$$E_i(f) = \frac{q_1^{i-s}(1-q_1)\left(d(q_1-q_0) + q_0^s\right)}{(1-q_1) + \left(1-q_1^{n-s}\right)(q_1-q_0)} \quad \text{for} \quad i = s,s+1,\ldots,n \quad ,$$

$$E_i(f) = \frac{q_0^{i-n}q_1^{n-s}(1-q_1)\left(d(q_1-q_0) + q_0^s\right)}{(1-q_1) + \left(1-q_1^{n-s}\right)(q_1-q_0)} \quad \text{for} \quad i = n+1,n+2,\ldots \quad .$$

How should the offense select his y_i -- that is, the probabilities of assigning exactly i waves to the attack? One reasonable criterion for him to use is to select the y_i so that the expected fraction of targets saved is a constant, no matter what n the defense assumes ($d \leq n \leq m$, where m denotes the maximum level of i). It turns out that the expected fraction of targets lost is

$$\sum_{i=1}^{m} E_i(f)y_i = \frac{q_0 + d(q_1-q_0)}{1 + (1-q_0)(m-1)} \quad ,$$

a constant independent of n, if the y_i are chosen so that

$$y_1 = y_2 = \ldots = y_{m-1} = \frac{1 - q_0}{1 + (1-q_0)(m-1)} \quad ,$$

$$y_m = \frac{1}{1 + (1-q_0)(m-1)} \quad ,$$

with m determined from the equation

$$\sum_{i=1}^{m} iy_i = a \quad .$$

Interestingly, this offense strategy can be recognized as the Matheson preallocation strategy for the offense when q_1 equals unity and h, the lowest attack level, is also unity. Note that the offense does not need to know the defensive value of q_1 in applying this strategy (except when calculating the expected fraction of targets saved).

The gain in using a damage assessment strategy can be determined by comparing the above formula giving the expected fraction of targets saved with the corresponding Matheson preallocation formula (as noted above, the Matheson preallocation results are unchanged when the hypothetical attack is used). Again, consider the example in which $d = 2$, $m = 4$, $q_1 = 1$ and $q_0 = 1/2$. The offensive level a is equal to $14/5$, corresponding to a strategy of $y_1 = y_2 = y_3 = 1/5$ and $y_4 = 2/5$. The expected fraction of targets saved is equal to $3/5 = 0.600$. If preallocation strategies are used, one has

$$x_0 = 3/10, \quad x_1 = x_2 = x_3 = x_4 = 3/20, \quad x_5 = 1/10 \quad ,$$

$$y_0 = 16/100, \quad y_1 = y_2 = y_3 = y_4 = 14/100, \quad y_5 = 28/100 \quad ,$$

and the expected fraction of targets saved is $29/50 = 0.580$. In other words, the gain achieved by using a damage assessment strategy is equal to 0.020.

4.6 ATTACKER-ORIENTED DEFENSE STRATEGIES

The preceding sections considered the gain in effectiveness if the defense could assess damage to its targets; the next sections, in contrast, assess the situation in which the defense is unable to predict which target an offensive weapon is aimed at before engaging it with a missile. One would certainly expect the result to be a loss in effectiveness over the optimal strategy making full use of all information. However, preallocation strategies do not use all information either, and are not very close to optimal when the defense is relatively strong. In such cases, various non-preallocation strategies, such as those discussed here and earlier, may be attractive.

If the defense cannot evaluate the attack, and if he is limited to one-on-one engagements, the best he can do is to assign missiles at random to the offensive weapons. If the offense knows that the defense is limited to this strategy, he should attack each target with a weapons (assuming a is an integer).

If $d \geq a$, the defense is quite effective, since every weapon is assigned a missile; the expected fraction of surviving targets is given by

$$E(f) = q_1^a \quad .$$

If $d < a$, each incoming offensive weapon has a chance of only d/a of being assigned a missile; the number of weapons directed at a given target which are actually intercepted is a random variable having a binomial probability density function. The expected fraction of surviving targets is

$$E(f) = \sum_{i=0}^{a} \binom{a}{i} \left(\frac{d}{a}\right)^i \left(1 - \frac{d}{a}\right)^{a-i} q_0^{a-i} q_1^i \quad .$$

Consider the following example: $d = 2$, $a = 3$, $q_1 = 1$, $q_0 = 1/2$. The expected fraction of targets saved using the weapon defense strategy is $125/216 = 0.579$. The expected fraction of targets saved using defense-last-move and offense-last-move strategies is 0.708 and 0.500, respectively. If neither side knows the other's allocation, and optimal preallocation strategies are used, one has

$$x_0 = 3/10, \quad x_1 = x_2 = x_3 = x_4 = 3/20, \quad x_5 = 1/10 \quad ,$$

$$y_0 = 1/10, \quad y_1 = y_2 = y_3 = y_4 = 3/20, \quad y_5 = 3/10 \quad ,$$

and the expected fraction of targets saved is 0.55.

Note that the weapon strategy is superior to the preallocation strategy despite the fact that the defense has less attack information. The reason for this is simple: the weapon strategy makes use of <u>all</u> the defensive missiles (even though less efficiently than in defense-last-move), but the preallocation defense strategy ends up with an expected fraction $53/200 = 0.265$ of the defensive missiles unused.

It is clear from the above example that a defense strategy which assigns missiles to weapons instead of targets is worth studying in its own right; it may be advantageous to use such a strategy even when it is possible for the defense to predict which target an offensive weapon is aimed at. In principle, one should be able to divide (a,d,q_0,q_1) space into two regions — in one a preallocation (that is, target-oriented) defense strategy is preferred, and in the other, a weapon (that is, attacker-oriented) defense strategy is preferred. However, it is clear that in the latter case, the defense can expect to do better yet. If the defense has chosen not to engage a weapon aimed at a given target, then (unless, perhaps, q_0 is small) he might as well not engage any other weapons aimed at the same target, saving missiles to be used more advantageously against other weapons.

Weapon defense strategies have been studied in some detail. Weapon defense strategies will be referred to as attacker-oriented defense strategies in the remainder of Section 4.6. Assume, as usual, that the offense has a stockpile of A weapons to be used against a set of T equal-valued targets, and the defense has a stockpile of D missiles to be used in their defense. Any missile can be used to intercept a weapon attacking any target. A target is destroyed by an unintercepted weapon with probability p, and a missile destroys the weapon it is directed against with probability ρ. Both offense and defense know the value of A, D, T, ρ and p. The offense wishes to minimize E(f), the expected fraction of targets saved, and the defense wishes to maximize this quantity.

Attacker-oriented defense strategies are most likely to be useful when the defense has at least as many missiles as the offense has weapons. (Certainly, preallocation strategies are not likely to be useful in such a case, unless ρ is rather small.) Therefore, the restriction that at most one missile attacks a weapon (usually assumed in preallocation strategies) will be removed in the rest of Section 4.6.

4.6.1 Neither Side Knows the Other's Allocation

Suppose that both sides must make their allocations in ignorance of the other's. In this case it can be seen that the optimum strategies for both are to allocate missiles and weapons in random fashion as uniformly as possible. No proof of this fact appears generally available, so one will be given here.

It will first be shown that the optimal allocation of offensive weapons is as uniform as possible. Assume the contrary, so that in the optimal allocation two targets receive m and n weapons respectively, where $n > m + 1$. Then the probability, X, that the first target survives is the probability that it survives all of the m offensive weapons directed against it. This value is unknown to the offense, since he does not know how the defense has assigned missiles to the weapons, but it has a definite value for any specific engagement. Let the probability that the second target survives be YZP, where Y is the probability that the target survives the first m weapons directed against it, Z is the probability it survives the next n - m - 1, and P is the probability it survives the last. Thus the two targets contribute an expected number of survivors of X + YZP.

Compare this allocation with that produced by reassigning to the first target the last weapon allocated to the second. This gives a more uniform allocation. The change in the expected number of targets surviving is

$$X + YZP - XP - YZ = (X-YZ)(1-P) \quad .$$

This change may be positive or negative. But note that the original allocation was just as likely to make the contribution of Y + XZP expected survivors. In this case the change in the expected number of targets surviving is

$$Y + XZP - YP - XZ = (Y-XZ)(1-P) \quad .$$

Since these two situations are equally likely, the above two expressions may be added:

$$(X-YZ)(1-P) + (Y-XZ)(1-P) = (X+Y)(1-Z\ (1-P) \geq 0 \quad .$$

This expression is nonnegative, so that on the average, the more uniform allocation leads to a better outcome for the offense. Note that in certain special situations in which some of the probabilities are 0 or 1, some non-uniform allocations may be optimal; but the uniform allocation will always be optimal as well.

The proof that the optimal defensive allocation is as uniform as possible is somewhat similar. Set $\tau = 1 - \rho$. Then the probability that a target survives attack by a weapon to which have been assigned m missiles is $1 - p\tau^m$. Assume that the optimum defensive allocation is non-uniform, so that two weapons are assigned m and n defensive missiles respectively, where $n > m + 1$. Suppose that it happens that the offensive allocation assigns both of these to the same target. Then the probability the target survives is $X(1-p\tau^m)(1-p\tau^n)$, where X gives the effect of any other weapons assigned to it. Then the change in the number of targets surviving if the defense had used the more uniform allocation (m+1,n-1) is

$$X(1-p\tau^m)(1-p\tau^n) - X(1-p\tau^{m+1})(1-p\tau^{n-1}) = -Xp(1-\tau)(\tau^m-\tau^{n-1}) \leq 0 \quad .$$

So in this case the defense prefers the more uniform allocation.

Now suppose that it happens that the offensive allocation assigns the two weapons to different targets. Then these two targets contribute an amount to the expected number of survivors equal to $X(1-p\tau^m) + Y(1-p\tau^n)$, where X and Y indicate the effect of the other weapons assigned to them. Analogously to the situation encountered before, the outcome $X(1-p\tau^n) + Y(1-p\tau^m)$ is equally likely. Thus the overall change produced by using the more uniform defensive allocation is measured by

$$\begin{aligned}&\left(X(1-p\tau^m) + Y(1-p\tau^n) + X(1-p\tau^n) + Y(1-p\tau^m)\right)\\ &-\left(X(1-p\tau^{m+1}) + Y(1-p\tau^{n-1}) + X(1-p\tau^{n-1}) + Y(1-p\tau^{m+1})\right)\\ &= -(X+Y)p(1-\tau)(\tau^n-\tau^{n-1}) \leq 0 \quad .\end{aligned}$$

Again, the defense prefers the more uniform allocation. Thus the optimal defensive allocation is as uniform as possible.

Calculating exactly the value of E(f) resulting from these optimal allocations is tedious but straightforward. Set $k = [D/A]$, $r = D - Ak$, $P_1 = p(1-\rho)^k$ and $P_2 = p(1-\rho)^{k+1}$. The attacker has $A - r$ superior weapons with target kill probability P_1, and r inferior weapons with target kill probability P_2, but of course he does not know which is which. Set $j = [A/T]$ and $s = A - Tj$. Recognizing that the hypergeometric distribution is involved, one finds that

$$E(f) = \frac{s}{T} S_1 + \left(1 - \frac{s}{T}\right)S_2 \quad ,$$

where

$$S_1 = \sum_{i=1}^{j+1} (1-P_1)^i(1-P_2)^{j-i+1}\binom{r}{j-i+1}\binom{A-r}{i}\bigg/\binom{A}{j+1}$$

and

$$S_2 = \sum_{i=1}^{j} (1-P_1)^i(1-P_2)^{j-i}\binom{r}{j-i}\binom{A-r}{i}\bigg/\binom{A}{j} \quad .$$

It would be of interest to determine the boundary of $(A/T, D/T, p, \rho)$-space on which the E(f) corresponding to the randomized attacker-oriented defense strategy and the appropriate offense strategy is equal to the E(f) when preallocation offense and defense strategies (as determined by Matheson) are used. Such information would enable one to make decisions about which type of strategies to use.

4.6.2 Offense Knows How Defense Will Assign All Missiles

Another interesting case is the offense-last-move version of the above, namely that in which the offense knows how the defense will assign missiles to each offensive weapon. This is a less likely situation than the previous one, but there is at least one rather plausible case where it could arise, namely when $D \geq A$ and the defense assigns missiles to the first D arriving weapons in a sequential attack.

As will be seen, the offensive allocation problem is rather involved, but the optimal defense is to allocate missiles as uniformly as possible. The proof is similar to that of the previous section.

Assume the contrary, so that in the optimal allocation, two weapons receive m and n missiles respectively, where $n > m + 1$. In the notation of the previous section, the probabilities that a target survives an attack by these weapons are $1 - p\tau^m$ and $1 - p\tau^n$ respectively. Compare this defensive allocation with one in which the two weapons in question receive $m + 1$ and $n - 1$ missiles. Consider the optimal offensive allocation against the latter defensive allocation.

If the two weapons are assigned to the same target in the optimal offensive allocation, the same calculation used in the previous section shows that the offense does better against the less uniform defensive allocation. Next suppose that the optimal offensive allocation against the more uniform defensive allocation assigns the two weapons to different targets. Then the expected number of survivors among the two targets can be written

$$X(1-p\tau^{m+1}) + Y(1-p\tau^{n-1}) \quad .$$

Because this is an optimal allocation, one may assume $X \geq Y$, for if $X < Y$, one has

$$\left(X(1-p\tau^{m+1}) + Y(1-p\tau^{n-1})\right) - \left(X(1-p\tau^{n-1}) + Y(1-p\tau^{m+1})\right)$$

$$= (Y-X)p(\tau^{m+1} - \tau^{n-1}) \geq 0 \quad .$$

But if $X \geq Y$ one has

$$\left(X(1-p\tau^{m}) + Y(1-p\tau^{n})\right) - \left(X(1-p\tau^{m+1}) + Y(1-p\tau^{n-1})\right)$$

$$= -p\tau^{m}(1-\tau)(X - Y\tau^{n-m-1}) \leq 0 \quad ,$$

so the defense prefers the more uniform allocation.

Thus, the defense should allocate missiles as uniformly as possible. The same is true of the offense, in a sense: however, it is difficult to specify what "as uniformly as possible" means. A fairly satisfactory algorithm is available if the defense uses his optimal allocation. The offense recognizes that he has two types of weapons to be assigned to targets — $A - (D-A[D/A])$ superior ones with a kill probability $P_1 = p(1-\rho)^{[D/A]}$ and $D - A[D/A]$ inferior ones with a kill probability $P_2 = p(1-\rho)^{[D/A]+1}$. The problem thus has an interesting equivalent — the allocation of missiles of two different types (say, different warhead sizes) in a no-defense situation. If he assigns m_i superior weapons and n_i inferior weapons to the ith target, the probability of survival of the target is equal to $S_i = (1-P_1)^{m_i}(1-P_2)^{n_i}$.

It can be shown that the minimum possible value of E(f) is achieved if all S_i are equal. In general, it is impossible to allocate m_i (summing to $A - (D-A[D/A])$) and n_i (summing to $D-A[D/A]$) so that all S_i are equal, and it is a complicated combinatorial problem to find that allocation of the m_i and the n_i leading to the smallest realizable value of E(f). The following algorithm for allocating the m_i and n_i has been proposed:

(1) sequentially assign each superior weapon to that target with the highest probability of survival S_i (taking all earlier weapons assignments to targets into account),

(2) after superior weapons have been exhausted, assign inferior weapons to targets in the same manner, and

(3) make pairwise comparisons between targets to see whether there exist exchanges of weapons which will further decrease E(f).

Burr and Graham (1970) have studied algorithms of this sort. They prove that the above algorithm leads to the optimum, provided that $\alpha = \log(1-P_2)/\log(1-P_1)$ is an irrational number. In fact, only step (3) is actually needed by the algorithm; steps (1) and (2) are intended to give a solution somewhere near the optimum before the repetitive step (3) is begun. It is also worth noting that in the optimal allocation at most three distinct pairs (m_i, n_i) can occur. The condition that α be irrational has the effect of eliminating so-called degeneracies, that is, cases in which there can exist two essentially different allocations giving the same value of E(f). Since α can easily be rational, it would be desirable to find a rule to handle degeneracies. One possibility would be to perturb α in some fashion.

If D/A is an integer, then the offense strategy is easy to specify — assign $[A/T]$ weapons to $T_1 = T - (A-T[A/T])$ targets, and $[A/T] + 1$ weapons to $T_2 = A - T[A/T]$ targets. The expected fraction of survivors if both sides use optimum strategies is

$$E(f) = (T_1/T)\left(1-p(1-\rho)^{D/A}\right) + (T_2/T)\left(1-p(1-\rho)^{(D/A)+1}\right) \quad .$$

One E(f) has been calculated, the weapons analyst can decide whether it is more advantageous for the defense to use an attacker-oriented strategy or a preallocation strategy. It would be of interest to determine the boundary on $(A/T, D/T, p, \rho)$-space on which the two strategies have the same E(f).

4.7 OFFENSIVE DAMAGE ASSESSMENT STRATEGIES

The preceding two sections discussed the modifications that must be made in both offensive weapon and defensive missile allocation strategies when the defense knowledge of the attack or its consequences is altered. This section considers the analogous problem for the offense. How much better can the offense do if he attacks in waves, and has some knowledge of the effectiveness of earlier waves before deciding which targets are to be attacked on the current wave? What must the defense do to counter such an offense?

4.7.1 Strategies if Targets Are Soft and Defensive Missiles Are Reliable

To be specific, assume that the offense attacks a set of T targets defended by a single missile stockpile. Suppose that the offense can determine whether or not each of his weapons has been attacked by a defensive missile. If one assumes that $q_1 = 1$ and $q_0 = 0$ (that is, a defensive missile always destroys an incoming weapon, and the target is destroyed by an unintercepted weapon), this knowledge is equivalent to knowing with certainty which targets have been destroyed on earlier waves. As before, let a denote the offensive weapon stockpile per target, and d ($\leq$ a) the defensive missile stockpile per target; both stockpiles are known to the offense and defense. Suppose that the offense is restricted to a k-wave attack; that is, on (k-1) occasions, he can evaluate the effect of earlier waves before assigning a new wave of weapons to targets. Finally, suppose that the offense wishes to maximize the expected fraction of targets destroyed and the defense wishes to minimize this quantity.

Goodrich (1967) derives the optimum offensive and defensive strategies to be used in the situation described above. At the ith wave, the offense allocates a_i weapons per target to each of the so-far-undestroyed targets ($\Sigma a_i = a$). Note that fractional allocations are permitted. If a fraction f_i of the targets remain undestroyed after the first (i - 1) waves, then the actual attack level per target attacked is a_i/f_i. Suppose that the defense notes that a_i weapons per target have been assigned on the ith wave, and fully defends a fraction d_i/a_i of these targets with an allocation of d_i weapons per target from its stockpile ($\Sigma d_i = d$). Clearly, f_i is equal to $(d_1/a_1)(d_2/a_2)\ \dots\ (d_{i-1}/a_{i-1})$, and the fraction of targets destroyed on the ith wave is $f_i(1-(d_i/a_i))$. Therefore, the fraction of targets destroyed in a k-wave attack is

$$E(f) = \sum_{i=1}^{k} \prod_{j=1}^{i-1} \left((d_j/a_j)\right)\left(1-(d_i/a_i)\right) = 1 - \prod_{i=1}^{k} (d_i/a_i) \quad .$$

In order to minimize E(f), the defense observes a_i and selects d_i so that $d_i/a_i = d/a$. The offense strategy is immaterial, and the fraction destroyed is $E^*(f) = \max_a \min_d E(f) = 1 - (d/a)^k$.

However, a better strategy is available to the defense. He can ignore the offensive allocation a_i, and select $d_i = dT/k$ missiles to be used in the defense of each wave. It is not difficult to see that the fraction of targets destroyed when this strategy is used is less than or equal to $E^*(f)$, with equality occurring only if the offense allocates aT/k weapons to each wave. In short, the defense can take advantage of any nonoptimum offensive strategy besides ignoring the size of the offensive allocation.

For example, let $T = 12$, $k = 2$, $d = 2$ and $a = 4$, and suppose the offense attacks with 72 weapons on the first wave and 24 on the second wave. The defense can counter this by allocating 36 missiles on the first wave, saving 6 of the targets, and 12 missiles on the second wave, saving 3 of these 6; the fraction destroyed is 3/4. However, a superior defense strategy consists of allocating 24 missiles on the first wave, saving 4 of the targets, and 24 missiles on the second wave, saving all of these 4 targets again; the fraction destroyed is 2/3. Of course, the offense can counter this allocation by assigning 48 weapons to the first wave and 48 to the second wave. Then, the defense can do no better than save 3 targets as before.

These results are directly applicable only when the number of offensive weapons and defensive missiles are integral multiples of the number of targets attacked and defended at each wave (as in the example). Goodrich (1967) describes how the offense and defense strategies must be modified if this is not the case — if different integral numbers of missiles or weapons must be assigned to different targets in a single wave of the attack.

4.7.2 Strategies if Targets Are Hard and Defensive Missiles Are Reliable

The preceding analysis assumed that q_0, the probability that the target survives an attack by an unintercepted weapon, is equal to zero. Suppose that q_0 is 1 - p instead. How does this modify the offensive and defensive strategies? This problem is much more difficult; in order to make any headway, Goodrich (1967) introduces the simplifying assumption that the offense does not reattack a target on future waves if a weapon assigned to that target was not intercepted by the defense (even though the target may survive). The expected fraction of targets destroyed on the ith wave becomes $f_i\left(1-(d_i/a_i)\right)\left(1-(1-p)^{a_i/f_i}\right)$, where the last term denotes the probability of target kill by (a_i/f_i) unintercepted weapons acting independently of each other. As before, the expected fraction of targets destroyed in a k-wave attack is given by

$$E(f) = \sum_{i=1}^{k} f_i\left(1-(d_i/a_i)\right)\left(1-(1-p)^{a_i/f_i}\right) \quad ,$$

where

$$f_i = \prod_{j=1}^{i-1} (d_i/a_i) \quad .$$

It appears impossible to solve in closed form for the offensive and defensive strategies satisfying $\max_a \min_d E(f)$, under the constraints that $\Sigma a_i = a$ and $\Sigma d_i = d$. Goodrich points out that dynamic programming can be used to obtain the optimum strategies if a computer is available. However, this is likely to be difficult, since dynamic programming tends to be unsuitable for max-min problems in practice.

However, an upper bound for $E^*(f) = \max_a \min_d E(f)$ can be easily obtained. Assume that an infinite number of waves can be assigned by the offense. Let the first wave attack be $a_1 = a - d$ weapons per target. If the defense responds with d_1 missiles per target, let the second wave attack be $a_2 = d_1$ weapons per target;

if the defense responds with d_2 missiles per target to the second wave, let the third wave attack be $a_3 = d_2$ weapons per target; and so on. The undefended targets on the first wave are attacked with a - d weapons apiece; the undefended targets on the second wave attacked with $d_1/f_1 = d_1(a-d)/d_1 = a - d$ weapons apiece; and so on. In an infinite-wave attack, every target is eventually attacked but not defended; hence, the expected fraction of targets destroyed is

$$E(f) = 1 - (1-p)^{a-d} \quad .$$

The same fraction of targets is destroyed if the attack sends in weapons one at a time and the defense engages the first dT attackers one-on-one; after the defense stockpile is exhausted, the offense allocates his remaining weapons evenly among the targets. It is impossible for the offense to do better than this.

Now let k again be finite, and suppose that the defense, instead of seeking the optimum strategy, adopts one of the strategies discussed previously — that is, suppose the defense observes a_i weapons per target on the ith wave and allocates d_i missiles per target so that $d_i/a_i = d/a = h$. Goodrich shows by the method of Lagrange multipliers that the optimum offense strategy corresponding to this defense strategy is to allocate the same number of weapons to each target being attacked:

$$a_1 = a_1, a_2 = a_1 h, a_3 = a_1 h^2, \ldots, a_k = a_1 h^{k-1} \quad ,$$

so that $a_1 = a(1-h)/(1-h^k)$. The expected fraction of targets destroyed is

$$E(f) = \left(1-(1-p)^{a(1-h)/(1-h^k)}\right)(1-h^k) \quad .$$

For example, if p = 1/2, k = 2, T = 9, a = 15/9, and d = 10/9, then the offense allocates 1 weapon per target to the first wave and 2/3 weapon per target to the second wave. However, since only 6 targets are attacked on the second wave, the offense allocates 1 weapon per target to the targets actually attacked. The defense strategy allocates 6/9 missiles per target on the first wave (fully defending 6 of the 9 targets), and 4/9 missiles per target on the second wave (fully defending 4 of the 6 surviving targets). The expected fraction of targets destroyed is E(f) = 5/18.

4.7.3 Strategies if Defensive Missiles Are Unreliable

It would be of interest to derive the optimum defense and offense strategies for a k-wave attack when the defensive missile reliability ρ is less than unity, for either $p = 1$ or $p < 1$. In order to make either problem analytically tractable it may be necessary to use as a criterion of effectiveness the expected number of weapons penetrating the defense (summed over all targets) instead of the expected fraction of targets destroyed. Problems such as these have been considered in Chapter 3.

More generally, suppose that the offense knows not only whether or not his weapon has been engaged (and, therefore, destroyed with probability $\rho = 1$ or $\rho < 1$), but also knows where the weapon has impacted with respect to the target. If a cookie-cutter damage-function is assumed, this is equivalent to assuming that the offense knows whether or not he has destroyed the target (and can therefore decide whether or not to include it in future attack waves); if a diffused Gaussian damage function is assumed, he can at least assess the probability of target kill (and take this into account in future attack waves). No work on this problem is known.

4.7.4 Damage Assessment for Unconstrained Offensive Weapon Stockpiles

Brodheim, Herzer and Russ (1967) discuss a somewhat different offensive damage assessment problem. They assume that the offense is attacking a set of T isolated point targets defended by a single stockpile of D defensive missiles. Each defensive missile has a probability ρ of destroying an offensive weapon, and each unintercepted weapon has a probability p of destroying the target it is aimed at.

The offense and defense both know the values of T, D, ρ and p; however, the defense does not know the size of the offensive stockpile. The offense assigns one weapon at a time to a target, and between firings can observe whether or not the target survived the weapon. The offense continues to fire weapons at undestroyed targets until he has destroyed all T targets.

Assume that the defense wishes to allocate defensive missiles to incoming weapons in such a way as to maximize the expected number of weapons required to destroy T targets. The optimum missile allocation can be found by means of a recursive argument (dynamic programming), working back from the end of the engagement to the start. Let $f(i,j)$ be the expected number of offensive weapons required to destroy T targets, given j defensive missiles available. The initial conditions are easily calculated:

$$f(0,j) = 0 \qquad \text{for all } j$$

$$f(i,0) = 1 + (1-p)f(i,0) + pf(i-1,0) \quad ,$$

so that $f(i,0) = i/p$. The recursive equation is:

$$f(i,j) = \max_{0 \le m \le j} \left(1 + f(i,j-m)\left(1-p(1-\rho)^{m}\right) + f(i-1,j-m)p(1-\rho)^{m}\right) \quad .$$

From this, one can iteratively find f(T,D) and the associated defense strategy with the aid of an electronic computer. One should note that this problem generalizes one considered in Section 3.2.4.

If this recursive equation is solved repeatedly for different values of i, one can find that value of i corresponding to a specified offensive stockpile A. The quantity i/T can then be regarded as an upper bound to the expected fraction of targets destroyed when the offensive damage assessment is imperfect, as assumed in the models discussed earlier in this section.

4.8 SUMMARY

This chapter, the core of the monograph, considers the joint offensive and defensive strategies when a salvo of identical weapons attacks a group of identical targets, any of which can be defended by any of a stockpile of identical unreliable missiles in one-on-one engagements. The criterion used throughout the chapter is the expected fraction of targets saved. Damage radius and aiming error, separately treated in earlier chapters, are subsumed into a single probability of kill of the target by an unintercepted weapon. The offense and defense know each other's stockpile sizes, the weapon reliability, and the missile unreliability; the models in this chapter differ principally in the amount of additional information each may have about the attack.

After a short discussion of defense-last-move (defense knows allocation of weapons to each target) and offense-last-move (offense knows allocation of missiles to each target) strategies, the problem of how each side preallocates when neither knows the other's allocation is discussed at considerable length. Less is known about a related class of strategies in which the offense preallocates normally but the defense allocates missiles to the defense of target groups instead of single targets.

The defense can save somewhat more targets if the offense attacks sequentially (one weapon per target in each wave) and the defense can assess damage to targets between weapon arrivals. On the other hand, if the defense cannot even determine which target a weapon is directed against at the time that an intercept must be decided upon, preallocation strategies for the defense are impossible and he must assign missiles as evenly as possible to weapons. Damage assessment is a two-way street, and the final section determines the advantages that accrue to the offense if he can assess target damage during the course of a sequential attack.

CHAPTER FIVE

OFFENSE AND DEFENSE STRATEGIES FOR A GROUP OF TARGETS WITH DIFFERENT VALUES

In the preceding chapter it was assumed that all targets in a group have the same value for both the defense and the offense. In this chapter, this requirement is relaxed — it is assumed instead that the ith target has a value v_i, a positive real number which is known to both the offense and defense. The question of how these values are agreed upon is not considered. If, for example, the targets to be defended are urban areas, the value may be proportional to population: if the targets to be defended are ICBM silos, the value may be proportional to the warhead yield or some related quantity.

In general, the mathematical model introduced at the beginning of Chapter 4 is carried over to multivalued targets. It is clear that offense and defense strategies will depend importantly upon the extent of knowledge the offense has about the defense, and vice versa. The criterion of effectiveness is now the expected value of the targets saved, rather than the expected fraction of the group. No restriction to one-on-one missile engagements is made in this chapter. However, in Sections 5.5.1-5.7.2, it is usually assumed that defensive missiles have perfect reliability so that a one-on-one missile defense is sufficient.

The definitions of offensive and defensive stockpiles must be slightly modified. In Chapter 4, most calculations were carried out on a per-target basis because all targets were identical; this is no longer possible. Instead, define a_i and d_i as the number of offensive weapons and the number of defensive missiles assigned to the ith target, respectively. The number of targets in the group is denoted by T. Let A and D denote the total stockpiles available for the engagement; thus,

$$A = \sum_{i=1}^{T} a_i \quad \text{and} \quad D = \sum_{i=1}^{T} d_i \ .$$

The constants A and D are assumed to be known to both the offense and the defense.

In Chapter 4, the quantities q_0 and q_1, defined as the survival probabilities of a target attacked by a single unintercepted weapon and a single intercepted weapon, respectively, were introduced.

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Both q_0 and q_1 were assumed to be constants known to the offense and the defense. In this chapter, q_0 is often regarded as a variable, denoted by $q_0(i)$, dependent upon the target being attacked. The reason for permitting q_0 to vary is twofold: if targets are to have varying values, there is usually no additional mathematical difficulty in permitting their other properties to vary, and it is reasonable that targets of different values have different vulnerabilities. For instance, a single unintercepted weapon will ordinarily destroy a much larger fraction of a city of 50,000 than a city of 1,000,000 population.

Because $q_1 = q_0 + \rho(1-q_0)$, where ρ is the defensive missile kill probability, it is clear that q_1 also depends upon the target attacked, and therefore is denoted by $q_1(i)$. Note that $1 \geq q_1(i) \geq q_0(i) \geq 0$.

The expected value surviving of a single target attacked by a_i weapons and defended by d_i missiles is

$$E(i) = v_i q_1(i)^{\min(a_i,d_i)} q_0(i)^{\max(0,a_i-d_i)} ,$$

and the expected value surviving of the group of targets is

$$E(V) = \sum_{i=1}^{T} E(i) .$$

The defense wishes to maximize E(V) and the offense wishes to minimize E(V) subject to their respective stockpile constraints, A and D. If there is no defense, $E(i) = v_i q_0(i)^{a_i}$.

The above criterion implies that only one-on-one missile engagements are allowed. Actually none of the optimization methods introduced in this chapter use this criterion in the above form. In Sections 5.2.1-5.3.2, the more general criterion $E(i) = E(i,a_i,d_i)$ is introduced; in Sections 5.4.1-5.4.2, a variant of E(i) eliminating the awkward exponents is defined; in Sections 5.5.1-5.6.4, the above model is specialized by assuming $q_0 = 0$ and $q_1 = 1$.

As might be anticipated, it is considerably more difficult to specify optimum offense and defense strategies for a group of targets of different value than it was for a group of targets of identical value. In general, the approaches that have been developed lead to approximate answers rather than exact ones. It is not always possible to prove that a given procedure will lead to the true optimum strategies; the reader is cautioned against an uncritical acceptance of the results.

5.1 OFFENSE ALLOCATION TO A GROUP OF TARGETS IN THE NO-DEFENSE CASE

Before examining more complicated models, it is worth considering the problem of optimum offensive allocation against a group of undefended targets having different values and different probabilities of kill. The offense wishes to allocate an integer valued set of weapons $(a_1, a_2, \ldots, a_T)$ so that the expected value saved of the group of targets,

$$E(V) = \sum_{i=1}^{T} E(i) = \sum_{i=1}^{T} v_i q_0(i)^{a_i} ,$$

will be minimized, subject to the condition that $a_i \geq 0$ and that he use no more weapons than are in his stockpile:

$$\sum_{i=1}^{T} a_i = A .$$

This problem has been recognized by missile system analysts for some time, and both exact and approximate solutions are available. It can be formulated as a transportation problem and solved by straightforward linear programming methods, as is done in den Broeder, Ellison and Emerling (1959), or it may be solved using an iterative procedure given by Manne (1958).

Both of the above methods yield integral answers, but require the use of a digital computer. If the number of weapons, A, is large with respect to the number of targets, T, somewhat simpler approximation methods can be used which yield nonintegral values for the a_i. A method using Lagrange multipliers is presented by Lemus and David (1963) and with more rigor by McGill (1970). Specifically, the offensive missile allocations are given as a function of the Lagrange multiplier, λ:

$$a_i^*(\lambda) = \log\left(-\lambda/v_i \log q_0(i)\right)/\log q_0(i) .$$

The value of λ is chosen (by trial and error, if necessary) to make the sum of the $a_i^*(\lambda)$ equal to A. If any of the individual a_i^* are negative, these targets are eliminated from consideration (they are not worth wasting offensive weapons on) and the process repeated on the reduced set.

Eventually one arrives at a set of positive a_i^* which are then rounded up or down to integer values. Dym and Schwartz (1969) have partially justified this heuristic procedure by showing that when all the $q_0(i)$ are equal, the optimal integer solution is attainable from the optimal continuous solution by a rounding procedure based on the fractional part of the continuous solution. In fact, when the $q_0(i)$ are equal, the problem becomes identical to that of Section 3.2.4.

Danskin (1967) obtains an approximate solution to the allocation problem by a slightly different approach. Specifically, he minimizes the function

$$E(V) = \sum_{i=1}^{T} E(i) = \sum_{i=1}^{T} v_i \exp(-b_i a_i),$$

subject to the constraints $a_i \geq 0$, $\Sigma a_i = A$. Here, the term $q_0(i)^{a_i}$ has been replaced by the exponential for analytic convenience. The constant b_i takes into account the fact that targets of different value have different resistances to destruction; $q_0(i) = \exp(-b_i)$.

To solve this problem, Danskin first proves a lemma which he attributes to the physicist J. Willard Gibbs:

> Suppose $\left(a_i = a_i^*,\ i = 1,2,\ldots,T\right)$ maximizes the function $\sum_{i=1}^{T} f_i(a_i)$ subject to the side conditions $\Sigma a_i = A$, $a_j \geq 0$. Suppose the f_i are all differentiable. Then there exists a constant λ such that $f_i'\left(a_i^*\right) = \lambda$ if $a_i^* > 0$ and $f_i'\left(a_i^*\right) \leq \lambda$ if $a_i^* = 0$.

In other words, the optimizing offense stra' ,y has the property that the slopes of the various $f_i'\left(a_i^*\right)$ with positive allocations are all equal — the "marginal utility" principle in economics. The Gibbs lemma is related both to the Lagrange multiplier principle and the Kuhn-Tucker conditions in the theory of mathematical programming.

Note that this lemma exhibits a property of optimum strategies; however, it does not say that any strategy having this property is therefore an optimal one. However, if there is only one strategy having the property, and the maximum is known to exist, then the strategy in question is indeed the maximizing one. (Of course it may be possible to determine in advance that the solution must be unique, for example if the functions f_i are convex.)

Danskin uses this fact to obtain the optimal continuous offense allocation for undefended targets. He proves that the offense attacks a target with

$$a_i^* = \frac{1}{b_i} \log_e \frac{v_i b_i}{\lambda}$$

weapons if $v_i b_i > \lambda$, and with $a_i^* = 0$ weapons if $v_i b_i < \lambda$. The value of the constant λ is determined by means of the constraint equation $\Sigma a_i^* = A$, using trial-and-error procedures. In short, the criterion for attacking or not attacking a target is given by the product $v_i b_i$: the higher this value, the more worthwhile it is for the offense to attack it.

In Chapter VI of his book, Danskin (1967) generalizes this method of allocating weapons to undefended targets. The offense is no longer restricted to attacking with a single stockpile of weapons having a specific yield; instead, he has W stockpiles of weapons, each with its own distinct yield. How should all these weapons be allocated to the targets to minimize the expected value surviving?

This problem is difficult to solve in general; Danskin outlines a method of solution for the following special case. Assume that the ith target is destroyed if a weapon from the jth stockpile lands within a distance R_{ij}, and is undamaged otherwise; assume also that weapons from the jth stockpile have an impact-point probability density function that is circular Gaussian, centered on the target with variance $\sigma_x^2 = \sigma_y^2 = \sigma_j^2$. Let a_{ij} denote the number of weapons from the jth stockpile allocated to the ith target, and let the stockpile sizes be A_j. Finally, let $a_{ij}/A_j = f_{ij}$. The problem is to find that set of f_{ij}, $i = 1,2,\ldots,T$ and $j = 1,2,\ldots,W$, which minimizes the surviving value

$$E(V) = \sum_{i=1}^{T} v_i \exp\left(-\sum_{j=1}^{W} f_{ij} A_j R_{ij}^2 / 2\sigma_j^2\right),$$

subject to the constraints $\sum_{i=1}^{T} f_{ij} = 1, \quad j = 1,2,\ldots,W.$

The solution is carried out iteratively, adding one weapon stockpile at a time. However, the solution is much simplified if the quantity $h_{ij} = A_j R_{ij}^2 / 2\sigma_j^2$ is separable; that is, if h_{ij} can be factored into the form $f(i)g(j)$. Fortunately, this is the case for one situation that is very plausible physically, namely if the squared distance R_{ij}^2 is given by the formula $b_i Y_j^{2/3}$, where b_i is a function of the resistance to destruction of the ith target, and Y_j is the yield of a weapon from the jth stockpile. Details of the separable solution are given on pp. 104-5 of Danskin (1967).

It is worth noting that Danskin's allocation of different weapon types against a set of undefended targets can sometimes be used to find the allocation of a single weapon type against a set of defended targets. Specifically, assume that the defense has D missiles and the offense has $A > D$ weapons available. Assume also that the offense assigns weapons to targets one at a time, and the defense assigns one missile apiece to the first D weapons which arrive. The first D weapons can be regarded as unengaged weapons with a reduced lethal radius (corresponding to a smaller probability of target kill); the final A - D weapons have their full lethal radius. In short, Danskin's method can be used to determine how the first D (engaged) weapons and the final A - D (unengaged) weapons should be allocated among targets having different hardnesses and values. This assumes that the resulting fractional allocations can be interpreted meaningfully. Note that if one demands that allocations be integral, this problem is the unequal-value analog of the problem of Section 4.6.2. Burr and Graham (1970) conjecture that a result similar to theirs may hold for the unequal-value case; but to prove it is likely to be difficult. In any case, dynamic programming could be applied to the problem.

5.2 TWO GENERAL TECHNIQUES FOR ONE-SIDED ALLOCATION PROBLEMS

The problem analyzed in the preceding section is an example of a one-sided allocation problem. For a specific defense, it is frequently not difficult to determine the optimum offense allocation, and for a specific offense, it is frequently not difficult to determine the optimum defense allocation, since these are effectively one-sided problems. This section introduces two general techniques suitable for such problems — dynamic programming and Lagrange multipliers. Note that the defense-last move and offense-last-move allocations introduced in Chapter 4 are not examples of one-sided allocation problems. A one-sided allocation technique cannot determine the best defensive allocation when the offense has the last move; it can only determine the best defense strategy to counter a specific offense strategy.

5.2.1 Dynamic Programming

The following description of dynamic programming is adapted from Bellman and Dreyfus (1962) and Young (1965). The methodology will be presented as the defense problem of maximizing the expected value saved against a given offense; the corresponding offense problem is similarly formulated. Let $E(i,d_i)$ be an arbitrary function denoting the expected value saved at the ith target if a defense d_i is allocated to it. It is convenient to generalize the constraint equation slightly. Let the cost of a defense missile allocated to the ith target be c_i, and assume that one can spend no more than C on defense missiles:

$$\sum_{i=1}^{T} c_i d_i \le C \quad .$$

Note that when all $c_i = 1$, this reduces to the usual stockpile constraint.

Dynamic programming solves the problem of maximizing $\Sigma E(i,d_i)$ subject to $\Sigma c_i d_i \le C$ by imbedding it in a two-dimensional family (k,R) of maximizations and solving these maximizations by recursive methods. Specifically, maximize

$$\sum_{i=1}^{K} E\left(i,d_i\right) = E_K\left(i,d_i\right); \qquad K = 1,2,\ldots,T \quad ,$$

subject to the constraints

$$d_i \ge 0 \ , \qquad \sum_{i=1}^{K} c_i d_i \le R \ , \qquad 0 \le R \le C \ ; \qquad K = 1,2,\ldots,T \quad .$$

Let max $E_K(i,d_i)$ be denoted by $h_K(R)$. The successive maximization problems can be solved by the recursive equation

$$h_K(R) = \max_{0 \le c_K d_K \le R} \left(E\left(K,d_K\right) + h_{K-1}\left(R-c_K d_K\right)\right) \ .$$

In other words, one first determines $h_1(R)$ for $0 \le R \le C$; then one uses these maximizations to determine $h_2(R)$ for $0 \le R \le C$; and so on until one finally obtains $h_T(C)$ and the associated d_i^*. To carry this out usually requires the aid of a digital computer. It is clear that if the c_i are all integers, it is sufficient to calculate $h_K(R)$ for integral values of R: $R = 1(1)C$. Young (1965) points out that the computation time is strongly affected by the choice of the c_i values; in general, it is advantageous to make the greatest common divisor of the c_i as large as possible. Thus, $c_1 = 100$, $c_2 = 300$, $c_3 = 400$ is much easier to work with than $c_1 = 104$, $c_2 = 301$, $c_3 = 397$. In many missile allocation problems, the c_i can be set equal to unity.

It should be pointed out that dynamic programming can be applied to considerably more general problems. For instance, the constraint can be replaced by $\sum_{i=1}^{T} c_i(d_i)$, where each c_i is an arbitrary function.

5.2.2 Lagrange Multipliers

The use of Lagrange multipliers for solving one-sided allocation problems is discussed in some detail by Everett (1963, 1965) and Charnes and Cooper (1965). Unlike the dynamic programming method, the Lagrange multiplier method does not necessarily lead to a maximizing defense allocation. However, it does guarantee that if any allocation at all is found, it will maximize the expected value of targets saved.

As before, let the cost of a defensive missile allocated to the ith target be c_i, and assume that one can spend no more than C on defense missiles:

$$\sum_{i=1}^{T} c_i d_i \le C \quad .$$

Let $E(i,d_i)$ be an arbitrary function denoting the expected value saved at the ith target if a defense d_i is allocated to it. The problem is to maximize $\Sigma E(i,d_i)$ subject to $\Sigma c_i d_i \le C$.

The following obvious but powerful lemma is a special case of Everett's main theorem concerning Lagrange multipliers.

> Let λ denote a positive real number.
> If $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ maximizes $\Sigma E(i,d_i) - \lambda \Sigma c_i d_i$ over all $(d_1, d_2, \ldots, d_T)$, then $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ maximizes $\Sigma E(i,d_i)$ over those $(d_1, d_2, \ldots, d_T)$ such that $\Sigma c_i d_i \le C^*$, where $C^* = \Sigma c_i d_i^*$.

In other words, if an unconstrained maximum of the Lagrangian $\Sigma E(i,d_i) - \lambda \Sigma c_i d_i$ can be found, then the maximizing $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ is also a solution to the constrained maximization problem. The unconstrained maximum might be found, for example, by differentiating the Lagrangian with respect to the d_i, setting these equations equal to zero, and solving for d_i. However, the differentiability of the Lagrangian is not a necessary requirement: all that is required is that it be possible to maximize the Lagrangian, by whatever means.

Often, the situation of interest in this chapter is that in which the d_i are restricted to be integers. The above lemma can be applied to this case either by making $E(i,d)$ a step function with the jumps at integral values, or by adding to the statement of the above lemma that the d_i are everywhere restricted to be integers. These two ways of viewing the situation are mathematically equivalent, but the latter is the more straightforward and will be the one used.

What value of λ should be taken? In general, different values of λ lead to different resource levels, C, and therefore it is necessary to solve the Lagrangian for various λ, trying to find one such that $\Sigma c_i d_i^* = C^*$.

If T is at all large, one might expect the many combinatorial possibilities to be a major stumbling-block in applying the Lagrangian method. However, the overall maximization can be achieved on a target-by-target basis; that is, it is sufficient to maximize each function $E(i,d_i) - \lambda c_i d_i$ separately. Of course, one must use the same value of λ in each of the T maximizations. If the Lagrangian for each target has been obtained, then the lemma guarantees that the result is a global maximum to the overall problem.

The primary difficulty with the Lagrangian method is the fact that in general it leads to solutions for certain isolated values of the resource level, C. If both C^* and the maximum $\Sigma E(i,d^*)$ are plotted as a function of λ, the curves resemble an irregular staircase. Each level part of the staircase corresponds to a given allocation $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$: the next step corresponds to an allocation in which one or more of the d_i^* have been increased.

If the staircase steps are small, the allocation corresponding to the actual stockpile constraint, C, is either the true maximizing strategy, or else is near enough to it for practical purposes. However, if the staircase steps are large, it will be a matter of luck whether the resulting value of C* is close enough to the desired value to be of use. Unfortunately, large jumps in C* represent the case most likely to occur in practice. However, there are some saving features. It can happen that some C* does come close to the desired value, and in any case the two values on either side of the desired C* give bounds on the solution. Moreover, it sometimes may be convenient to find an approximate solution from a nearby Lagrangian solution. Briefly, one perturbs the allocation $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ into $\left(d_1^* + \epsilon_1, d_2^* + \epsilon_2, \ldots, d_T^* + \epsilon_T\right)$ using integer ϵ_i, so that $\Sigma\left(d_1^* + \epsilon_i\right)c_i \leq C$; Everett calls $\Sigma|\epsilon_i|$ the ϵ-depth.

Sometimes, it is sufficient to substitute strategies having a small ϵ-depth (say, less than five or so) into $\Sigma E(i,d_i)$ to locate a larger expected value of targets saved, corresponding to a larger value of C. In short, the Lagrange method always yields a lower bound to the optimum allocation, and further trial and error on the Lagrangian will usually yield a better value.

5.3 GENERAL METHODS FOR CONSTRUCTING TWO-SIDED OFFENSE-LAST-MOVE STRATEGIES

This section, and the two sections following, examine a variety of methods for determining offense and defense strategies when the offense has the last move — that is, can make his own allocation after observing the defense allocation. As mentioned earlier, a variety of different criteria of loss have been used in order to make this difficult problem more tractable.

5.3.1 A Lagrangian Approach to Max-Min Problems

Pugh (1964) uses an arbitrary criterion $E(i,a_i,d_i)$; unfortunately, his heuristic method gives incorrect defensive allocations (in general), but often gives correct offensive allocations in response to the resulting defense allocations. However, Pugh is able to calculate approximate upper and lower bounds for the expected value of targets saved if optimum strategies are used by both sides. If the upper and lower bounds are close to each other, then Pugh's strategies (which correspond to the lower bound in terms of expected value of targets saved) can be used with assurance that they are approximately optimum.

Specifically, let $E(i,a_i,d_i)$ denote the expected value saved at the ith target if it is defended by d_i missiles and attacked by a_i weapons. Assume also that the total weapon stockpile is A and the defensive missile stockpile is D. If the offense can allocate weapons after observing the defensive allocation of missiles to targets, then the expected value saved is given by

$$E^*(V) = \max_{d} \min_{a} \sum_{i=1}^{T} E(i, a_i, d_i) ,$$

where the maximum is taken over all defense strategies such that $\Sigma d_i \leq D$, and the minimum over all offense strategies such that $\Sigma a_i \leq A$.

For any given defense strategy, the Lagrangian method introduced in the preceding section will give a bound on the optimum offense strategy $(a_1^*, a_2^*, \ldots, a_T^*)$. In other words, it is not difficult to obtain a lower bound to the expected value saved, $E^*(V)$. Pugh suggests using the Lagrangian method to obtain both offense and defense strategies. Specifically he introduces the function

$$L(\lambda, \omega) = \max_{d} \min_{a} \left(\sum_{i=1}^{T} E(i, a_i, d_i) - \lambda \sum d_i + \omega \sum a_i \right) .$$

For any assumed value of λ and ω, it is not in general difficult to find an unconstrained maximin of the bracketed expression, since the maximin can be calculated on a target-by-target basis, just as in the preceding section. Difficulty will be encountered only when a_i and d_i can range over a rather large set of values. By trial and error, one may be able to find that (λ, ω) pair which leads to a minimizing $(a_1^*, a_2^*, \ldots, a_T^*)$ and a maximizing $(d_1^*, d_2^*, \ldots, d_T^*)$ such that $\Sigma d_i^* = D$ and $\Sigma a_i^* = A$. Pugh proposes that these strategies be taken as the optimum strategies in the original problem. Of course, it is even less likely than in the one-sided case that one can find such λ and ω.

Even if such λ and ω can be found, there is no mathematical justification for this procedure. Although the offensive allocation that results will be an optimal response to the resulting defensive allocation, it is possible to construct examples in which the defensive allocation is not optimal. (However, this does lead to a lower bound on the expected value saved, $E^*(V)$, since the offensive allocation is optimal.)

Nevertheless, Pugh observes that in many practical problems and simple trial cases the strategies obtained by the Lagrangian turned out to be correct. For the missile analyst who wishes more than empirical justification, Pugh introduces a method for calculating approximate upper and lower bounds to the expected value saved, $E^*(V)$, when the correct strategies are used. If the difference between these bounds is small, then the use of the Lagrangian strategies (which lead to the lower bound of $E^*(V)$) is practicable. It seems plausible that the difference will be small whenever one has many targets of different values; a precise statement of such a criterion would be useful to have.

A lower bound to $E^*(V)$ is given by $\Sigma E\left(i,a_i^*,d_i^*\right) - \lambda_0 \Sigma d_i^* + \omega_0 \Sigma a_i^*$, where $\left(\lambda_0,\omega_0,a_1^*,a_2^*,\ldots,a_T^*,d_1^*,d_2^*,\ldots,d_T^*\right)$ is a maximin solution to the Lagrangian, and $\Sigma d_i^* = D$, $\Sigma a_i^* = A$. (If these constraints cannot be satisfied using the Lagrangian approach, they, too, must be bounded.) An upper bound to $E^*(V)$ is somewhat more tedious to calculate, as it involves finding maximin solutions to the Lagrangian for other values of λ and ω (leading to strategies with weapon stockpiles not equal to D). For example, assume that one has found the maximin solution $\left(\lambda_1,\omega_1,a_1',a_2',\ldots,a_T',d_1',d_2',\ldots,d_T'\right)$ to the Lagrangian, and that $\Sigma a_i' = A'$, $\Sigma d_i' = D'$. If

$$E\left(i,a_i',d_i'\right) - \lambda_1(A-A') + \omega_1(D-D') \leq \sum E\left(i_i,a_i^*,d_i^*\right) ,$$

then Pugh shows that one can eliminate a range of λ, $\lambda_1 \leq \lambda \leq \lambda_1'$, where λ_1' substituted for λ_1 in the above equation changes the inequality. If there are enough different Lagrangian solutions available, Pugh claims that only a small region of λ in the vicinity of λ_0 will not be eliminated — say, $\lambda_L \leq \lambda_0 \leq \lambda_H$. One can determine an upper bound for $E^*(V)$: the maximum value of $L(\lambda,\omega)$ in the region $\lambda_L \leq \lambda \leq \lambda_H$, $\omega = \omega_0$.

Note that there is no guarantee that this method will yield reasonably close upper and lower bounds to $E^*(V)$, no matter how many maximin solutions to the Lagrangian are calculated. Unfortunately, one does not know whether Pugh's method will succeed or fail until after considerable work has been done.

There exist cases in which the situation is more satisfactory. Penn (1971) considers such a case; see Section 5.6.4 of this monograph for details. Pearsall (1971) considers the possibility of bounding the solution of general max-min or min-max problems of the above sort by arbitrarily introducing mixed strategies for the offense, retaining the Lagrangian approach. This transformed problem may still not have a satisfactory solution, but Pearsall derives a set of complicated conditions under which it does. Unfortunately, the details are too intricate to give here.

5.3.2 A Dynamic Programming Approach to Max-Min Problems

Pugh attempted to apply the method of Lagrange multipliers to derive optimum offense and defense strategies in the offense-last-move problem. In contrast, Randolph and Swinson (1969) apply the method of dynamic programming to this problem. It is not too surprising to learn that they, like Pugh, run into fundamental mathematical difficulties; dynamic programming can no more be rigorously used than Lagrange multipliers. Instead, they end up by deriving upper and lower bounds to the expected value saved.

Specifically, Randolph and Swinson form the dynamic programming recursion relation

$$c_i(A,D) = \max_{0 \le a_i \le A} \; \min_{0 \le d_i \le D} \left\{ E(i,a_i,d_i) + c_{i-1}(A-a_i, D-d_i) \right\} .$$

Starting with i equal to unity, one solves iteratively for the optimizing values $a_1^*, d_1^*, a_2^*, d_2^*, \ldots, a_T^*, d_T^*$ and the corresponding $c_i(A,D)$. One might be tempted to interpret $c_i(A,D)$ in the following way: no matter what the policy might be for the last T - i pairs (a_i,d_i), the policy over the first i pairs (a_i,d_i) will be maximin. However, this is not necessarily true for $i \ge 3$; in fact, it is possible to obtain different values of $c_i(A,D)$ for different permutations of the targets. The difficulty, of course, occurs because insufficient information is stored in the backwards induction process of dynamic programming. In fact, the final $c_T(A,D)$ obtained by this iteration will be only an upper bound to the expected value of the total engagement, $E^*(V)$.

On the other hand, Randolph and Swinson show that if one adopts the offensive allocation $\left(a_1^*, a_2^*, \ldots, a_T^*\right)$ derived by this process, and then uses dynamic programming to determine that defensive allocation $\left(d_1', d_2', \ldots, d_T'\right)$ corresponding to this nonoptimal offensive allocation, then

$$E'(V) = \min_a \sum_{i=1}^{T} E\left(i, a_i^*, d_1'\right)$$

is a lower bound to the expected value $E^*(V)$. To solve the offense-last-move problem, then, Randolph and Swinson suggest calculating $(c_T(A,D), E'(V))$ for various random permutations of the targets, and stopping when the minimum $c_T(A,D)$ is acceptably near the maximum $E'(V)$. It is hard to say how many permutations need be selected to obtain a given accuracy; the authors demonstrate in a three-target problem (for which there exist only three different permutations to be tried) that min $c_T(A,D)$ is about 0.3 percent above max $E'(V)$.

There is a surprising philosophical similarity between Pugh on the one hand and Randolph and Swinson on the other. Both lead to upper and lower bounds bracketing the true $E^*(V)$; the quality of the results (i.e., the speed of convergence) is in general unknown for either one. In general, it is probably more worthwhile to seek more explicit optimization methods based on specific properties of $E(i,a_i,d_i)$. This will be the approach taken in the next two sections.

5.4 TWO-SIDED OFFENSE-LAST-MOVE STRATEGIES USING A SPECIAL PAYOFF FUNCTION

In Pugh (1964), the expected value surviving of a group of targets was specified only in general terms: $\Sigma E(i,a_i,d_i)$. It seems reasonable to expect that at least some of the mathematical difficulties encountered by Pugh might be avoided by restricting oneself to explicit $E(i,a_i,d_i)$. One such function was partially defined at the beginning of this chapter:

$$E(i) = v_i q_1(i)^{\min(a_i,d_i)} q_0(i)^{\max(0,a_i-d_i)} ,$$

where $q_0(i)$ is the probability of target survival (or expected fraction of target surviving) against an unengaged weapon directed against it, and $q_1(i)$ is the analogous probability against an engaged weapon. Both are dependent on the target considered and require further specification.

5.4.1 A Partial Solution in an Idealized Case

Unfortunately, it is quite difficult analytically to handle exponents of the form given above. To get around this difficulty, one can approximate the actual defense by an idealized defense in which each offensive weapon is intercepted by d_i/a_i defensive missiles. Because d_i/a_i is in general not an integer, it is convenient to express $E(i)$ in terms of exponentials rather than powers of $q_j(i)$:

$$E(i) = v_i\left(1 - s_i \exp(-t_i d_i/a_i)\right)^{a_i} = v_i u_i^{a_i} .$$

The term $\exp(-t_i d_i/a_i)$ is the probability that an individual offensive weapon penetrates the defense; t_i is related to the defensive missile reliability ρ_i by $(1-\rho_i) = \exp(-t_i)$. The term s_i is the probability that an unintercepted weapon will destroy the target (that is, $q_0(i)$). The quantity in parentheses (denoted by u_i) is the probability that the target survives an attack of one weapon; this is raised to a power to yield the probability that it survives an attack by a_i weapons.

The problem to be solved can now be stated quite simply. Assume that the offense weapon stockpile is A and the defensive missile stockpile is D. If the offense can allocate weapons after observing the defensive allocation of missiles to targets, the expected value of targets saved is given by

$$E^*(V) = \max_d \min_a \sum_{i=1}^{T} E(i) ,$$

where the maximum is taken over all defense strategies such that $\Sigma d_i \le D$, and the minimum over all offense strategies such that $\Sigma a_i \le A$. What offense strategy $\left(a_1^*, a_2^*, \ldots, a_T^*\right)$ and defense strategy $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ yields $E^*(V)$?

Danskin (1966, 1967) does not attempt to find the optimizing strategies; apparently this is a very difficult analytic problem. Instead, he considers a more restricted problem — that of obtaining partial criteria for deciding whether or not a target should be defended at all. He introduces two formulas:

$$C_i = v_i \log \frac{1}{1 - s_i} ,$$

$$D_i = \frac{t_i s_i}{\left(1-s_i\right) \log \frac{1}{1 - s_i}} .$$

For each target, the number-pair (C_i, D_i) is readily calculated. Danskin proves the following lemma:

> Assume that the ith target is defended in an optimum strategy.
>
> 1. If $C_j \ge C_i$ and $D_j \ge D_i$, the jth target will also be defended in the optimum strategy.
> 2. If $C_j \ge C_i$ and $D_j < D_i$, the jth target may or may not be defended in the optimum strategy; if it is not defended, the optimum attack strategy hits both targets i and j.

3. If $C_j < C_i$ and $D_j \geq D_i$, the jth target may or may not be defended in the optimum strategy; if it is not defended, the optimum attack strategy hits target i but not target j.

If $C_j < C_i$ and $D_j < D_i$, no conclusion can be drawn. Note that if the C_i and D_i are ordered in the same way, only the first conclusion applies; that is, the optimum defense is one in which all targets having values of C_i above a given constant C_0 are defended. This situation occurs if v_i varies but s_i and t_i are constant over targets, or if t_i varies but s_i and v_i are constant over targets.

5.4.2 An Approximate Solution in a Limiting Case

Obviously, Danskin's result is not of much use to the missile defense designer; he does not know how much defense to assign to each target. Perkins (1961), however, is able to determine an approximation to the optimum defense and offense strategies yielding E*(V). He succeeds in expressing the solution in closed form only in a limiting situation — when the ratios A/D and A/T are both quite large. In addition, he allows fractional allocations. All targets must be attacked quite heavily; to be specific, a_i^* must exceed a_i^o, where a_i^o is the solution to the equation

$$\left(v_i - E\left(i,a_i^o,d_i^*\right)\right)/a_i^o = -\frac{\partial}{\partial a_i} E\left(i,a_i,d_i\right)\Bigg|_{\substack{d_i=d_i^* \\ a_i=a_i^o}} .$$

If any of the offensive allocations a_i^* are less than a_i^o, then it will pay the offense to borrow weapons from other targets (or, if such borrowing lowers other a_j^* below a_j^o, refraining from attacking some targets) in order to build the attack up to a_i^o. Discontinuities such as these in the offense strategy cause the closed-form solution of Perkins to break down.

Perkins first proves that defense and offense allocations are always proportional to each other in the closed form solution:

$$d_1^*/a_1^* = d_2^*/a_2^* = \ldots = d_T^*/a_T^* = D/A .$$

This being so, it is sufficient to find the offense strategy as a function of v_i. For the ith target, the offense assigns a_i^* weapons:

$$a_i^* = \left(\log_e C - \log_e\left(-v_i \log_e u_i\right)\right)/\log_e u_i ,$$

where u_i is the probability that the target survives an attack of one weapon, defined earlier as $\left(1 - s_i \exp(-t_i d_i/a_i)\right)$. (Perkins actually derives the offensive allocation for an arbitrary function $f_i(d_i/a_i)$ instead of the specific exponential $\exp(-t_i d_i/a_i)$.) Note that the ratio d_i/a_i must be replaced by D/A in order to compute u_i. The constant C is found with the aid of the offensive stockpile constraint A:

$$\log_e C = \frac{A + \sum_{i=1}^{T} \log_e\left(-v_i \log_e u_i\right)/\log_e u_i}{\sum_{i=1}^{T} \left(1/\log_e u_i\right)} .$$

Also, Perkins shows that

$$E^*(V) = \sum_{i=1}^{T} v_i u_i^{a_i^*} = C \sum_{i=1}^{T} 1/\left(-\log_e u_i\right) .$$

Perkins presents a lengthy scheme for insuring that a valid solution is obtained; the reader is referred to his paper for details. To determine whether or not the closed-form solution given above is valid, one first observes whether or not any of the a_i^* are negative. If so, these targets are eliminated from the set (that is, they are left undefended) and the closed-form solution is derived for the remaining targets. When all a_i^* are positive, a final check is necessary. The number of offensive weapons assigned to the ith target, a_i^*, must satisfy the following inequality (calculated from equations (38), (24) and (22) in Perkins):

$$a_i^* \log_e u_i + u_i^{-a_i^*} < 1 + t_i d_i \left(1-u_i\right)/u_i .$$

If this inequality holds for all i such that $a_i^* > 0$, then the closed-form solution is valid. When this inequality does not hold, further (more complicated) checks are required.

5.5 TWO-SIDED OFFENSE-LAST-MOVE STRATEGIES FOR RELIABLE MISSILES

In this section, the expression giving the expected value of targets saved is drastically simplified. Assume that an unintercepted offensive weapon destroys the target at which it is aimed with probability p, and that an intercepted offensive weapon is itself destroyed with probability one. All engagements are therefore one-on-one, and

$$E(i) = v_i(1-p)^{\max(0,a_i-d_i)} .$$

The expected value saved of the group of targets is

$$E(V) = \sum_{i=1}^{T} E(i) .$$

5.5.1 A Lagrangian Approach to a Specific Max-Min Problem

Eisen (1967) attempts to employ the heuristic Lagrangian approach of Pugh (1964), discussed in Section 5.3.1, to find plausible solutions of the above problem (fractional allocations being permitted). Specifically, he tries to find the unconstrained (except for the obvious requirement that all d_i and a_i be nonnegative) max-min

$$\max_{d_i} \min_{a_i} \left(E(i) - \lambda d_i + \omega a_i\right) .$$

Unfortunately, his analysis is incomplete, leading to a restricted set of solutions; therefore, a complete solution will be sketched below.

It is fairly easy to determine the inner minimization. Set

$$x = -\log_e(1-p) ,$$

$$t_i = (\omega/x)\left(1 - \log_e\left(\omega/v_i x\right)\right) .$$

It can be seen that $t_i \le v_i$ in all cases. Dropping the subscript i for convenience and setting $E = E(i)$, the inner minimization can (after some algebra and differential calculus) be written in two forms:

1. $\min_a (E - \lambda d + \omega a) = v - \lambda d$ if $\omega \ge vx$ or if $d \ge (v-t)/\omega$, in which case $a = 0$ yields the minimum.

2. $\min_a(E - \lambda d + \omega a) = t - \lambda d + \omega d$ if $\omega \leq vx$ and $d \leq (v-t)/\omega$, in which case $a = d - \log_e(\omega/vx)/x$ yields the minimum.

If $\omega \leq vx$ and $d = (v-t)/\omega$, the two values are the same, and the offense can select either $a = 0$ or $a = d - \log_e(\omega/vx)/x$ at will.

From the above, one sees that the value of the inner minimum is a piecewise linear function of d with at most two pieces. This fact makes it fairly easy to find the optimizing values of d and a for any λ and ω. There are three cases:

1. $\lambda < \omega$. Here $d = (v-t)/\omega$, provided $\omega < vx$. One may choose a to be either 0 or $d - \log_e(\omega/vx)/x = (v/\omega) - (1/x)$ at will. If $\omega \geq vx$, $d = a = 0$.
2. $\lambda = \omega$. Here one may choose <u>any</u> d in the range

   $0 \leq d \leq (v-t)/\omega$, provided $\omega < vx$. If $d = (v-t)/\omega$, one may again choose a to be either 0 or $(v/\omega) - (1/x)$ at will. If $d < (v-t)/\omega$, then $a = d - \log_e(\omega/vx)/x$. As before, if $\omega \geq vx$, $d = a = 0$.
3. $\lambda > \omega$. In this case $d = 0$ always, and $a = \max(0, -\log_e(\omega/vx)/x)$.

These solutions give the offense and defense allocations at a single target of value v. As noted in earlier sections, one attempts to select λ and ω by trial and error so that $\Sigma a_i = A$, $\Sigma d_i = D$.

Eisen's solution corresponds to case (1) with the additional restriction that $a = (v/\omega) - (1/x)$ whenever possible. It is of interest that λ functions only as a switch among the three cases. If one wishes to allocate missiles and weapons from a pair of stockpiles A and D, case (2) gives the widest range of possibilities, since (2) includes the allocations of (1), and (3) is trivial. The fact that d at a target can often be chosen anywhere in some range, even for ω fixed, makes it likely that it will often be possible to match A and D rather well by taking $\lambda = \omega$. It would be of great interest to know how the results of this heuristic method compare with the actual constrained overall maximin.

5.5.2 A Special Case: Reliable Weapons

Suppose now that an unintercepted offensive weapon destroys the target at which it is aimed with probability one, and that an intercepted weapon is destroyed with probability one. Then

$$E(i) = v_i \quad \text{if} \quad a_i \leq d_i ,$$
$$= 0 \quad \text{if} \quad a_i > d_i .$$

The expected value surviving of the group of targets is

$$E(V) = \sum_{i=1}^{T} E(i) \ .$$

The problem to be solved is the following. Assume that the offensive weapon stockpile is A and the defensive missile stockpile is D. If the offense can allocate weapons after observing the defensive allocation of missiles to targets, the expected value of targets saved is given by

$$E^*(V) = \max_{d} \min_{a} \sum_{i=1}^{T} E(i) \ ,$$

where the maximum is taken over all defense strategies such that $\Sigma d_i \le D$, and the minimum over all offense strategies such that $\Sigma a_i \le A$. What offense strategy $\left(a_1^*, a_2^*, \ldots, a_T^*\right)$ and defense strategy $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ yields $E^*(V)$?

This problem has been treated by van Lint and Pollak (1972). Specifically, they assume an offensive stockpile size normalized to unity and a defensive stockpile size of $H = D/A$. The admissible offense strategies are real numbers $(a_1, a_2, \ldots, a_T)$ such that $\Sigma a_i = 1$; the admissible defense strategies are real numbers such that $\Sigma d_i = D/A$. In an actual allocation, the a_i and d_i must, of course, be integers; therefore, the strategies derived by van Lint and Pollak are limiting ones. It is possible to estimate the error involved when A and D are small; hence the results may be useful then, too.

Using techniques from the theory of linear equations and number theory, van Lint and Pollak show that there are certain canonical defense strategies corresponding to defense stockpiles $H_1, H_2, \ldots, H_K$. If the actual defense stockpile size is H, the defense can achieve the same expected target value saved by using only H_i defensive missiles, where $H_i \le H \le H_{i+1}$. In other words, it is possible to list the complete set of optimum offense and defense strategies for $1 \le D/A \le T$; when $D/A > T$, a perfect defense is possible, and when $D/A < 1$, no defense is possible. The offense strategy actually selected sometimes depends upon the relative target values; however, it is an easy matter to check through the admissible offense strategies to find that one which minimizes the expected value of targets saved.

The table below indicates all possible optimum defense and offense strategies as a function of H = D/A, for targets of arbitrary value in groups of two through five. The targets are listed in descending order of value: $v_1 \geq v_2 \geq v_3 \geq v_4 \geq v_5$. The defense strategies are self-explanatory; the offense attacks the indicated value points with enough force to overwhelm the defense there. When there are two or more entries in the offense column, the offense chooses the one that maximizes the value destroyed. When two or more defense strategies are listed with a given D/A, the defense chooses the one that minimizes the maximum value destroyed.

Two Targets

Defense Strategy $\left(d_1^*, d_2^*\right)$	D/A Required	Value Destroyed by Offense
(1,0)	1	v_2
(1,1)	2	None

Three Targets

Defense Strategy $\left(d_1^*, d_2^*, d_3^*\right)$	D/A Required	Value Destroyed by Offense
(1,0,0)	1	(v_2, v_3)
(1/2,1/2,1/2)	3/2	v_1
(1,1,0)	2	v_3
(1,1,1)	3	None

Four Targets

Defense Strategy $\left(d_1^*, d_2^*, d_3^*, d_4^*\right)$	D/A Required	Value Destroyed by Offense
(1,0,0,0)	1	(v_2, v_3, v_4)
(1/3,1/3,1/3,1/3)	4/3	(v_1, v_2)
(1/2,1/2,1/2,0)	3/2	(v_1, v_4)
(2/3,1/3,1/3,1/3)	5/3	v_1 or (v_2, v_3)
(1/2,1/2,1/2,1/2)	2	v_1

Four Targets (continued)

Defense Strategy $(d_1^*, d_2^*, d_3^*, d_4^*)$	D/A Required	Value Destroyed by Offense
(1,1,0,0)	2	(v_3, v_4)
(1,1/2,1/2,1/2)	5/2	v_2
(1,1,1,0)	3	v_4
(1,1,1,1)	4	None

Five Targets

Defense Strategy $(d_1^*, d_2^*, d_3^*, d_4^*, d_5^*)$	D/A Required	Value Destroyed by Offense
(1,0,0,0,0)	1	(v_2, v_3, v_4, v_5)
(1/4,1/4,1/4,1/4,1/4)	5/4	(v_1, v_2, v_3)
(1/3,1/3,1/3,1/3,0)	4/3	(v_1, v_2, v_5)
(2/5,2/5,1/5,1/5,1/5)	7/5	(v_1, v_2) or (v_1, v_3, v_4)
(1/2,1/2,1/2,0,0)	3/2	(v_1, v_4, v_5)
(1/2,1/4,1/4,1/4,1/4)	3/2	(v_1, v_2) or (v_2, v_3, v_4)
(3/5,2/5,1/5,1/5,1/5)	8/5	(v_1, v_3) or (v_2, v_3, v_4)
(2/3,1/3,1/3,1/3,0)	5/3	(v_1, v_5) or (v_2, v_3, v_5)
(1/3,1/3,1/3,1/3,1/3)	5/3	(v_1, v_2)
(3/4,1/4,1/4,1/4,1/4)	7/4	v_1 or (v_2, v_3, v_4)
(1/2,1/2,1/4,1/4,1/4)	7/4	(v_1, v_3) or (v_3, v_4, v_5)
(3/5,2/5,2/5,1/5,1/5)	9/5	(v_1, v_4) or (v_2, v_3) or (v_2, v_4, v_5)
(1,1,0,0,0)	2	(v_3, v_4, v_5)
(1/2,1/2,1/2,1/2,0)	2	(v_1, v_5)
(2/3,1/3,1/3,1/3,1/3)	2	v_1 or (v_2, v_3)

Five Targets (continued)

Defense Strategy $(d_1^*, d_2^*, d_3^*, d_4^*, d_5^*)$	D/A Required	Value Destroyed by Offense
(3/4,1/2,1/2,1/4,1/4)	9/4	v_1 or (v_2, v_4)
(1,1/3,1/3,1/3,1/3)	7/3	(v_2, v_3)
(2/3,2/3,1/3,1/3,1/3)	7/3	v_1 or (v_3, v_4)
(1,1/2,1/2,1/2,0)	5/2	(v_2, v_5)
(1/2,1/2,1/2,1/2,1/2)	5/2	v_1
(1,2/3,1/3,1/3,1/3)	8/3	v_2 or (v_3, v_4)
(1,1,1,0,0)	3	(v_4, v_5)
(1,1/2,1/2,1/2,1/2)	3	v_2
(1,1,1/2,1/2,1/2)	7/2	v_3
(1,1,1,1,0)	4	v_5
(1,1,1,1,1)	5	None

Unfortunately, the number of combinatorial possibilities goes up rapidly with the number of targets, so this approach is feasible only for very small groups. It is also worth noting that the methods of van Lint and Pollak produce all possible optimum defense strategies, but in addition may produce some nonoptimum defense strategies. Although none of the defense strategies in the above tables is nonoptimum, it can be demonstrated that (for example) the defense strategy (2/3,2/3,2/3,2/3,1/3,1/3,1/3), corresponding to D/A = 11/3, is inferior to the defense strategy (1/2,1/2,1/2,1/2,1/2,1/2,1/2), corresponding to D/A = 7/2, for all possible vectors $(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$.

An important practical criticism of the van Lint and Pollak defense strategies is that they depend upon an exact knowledge of the offensive stockpile size, A. It is clear from an examination of the optimum defense strategies that if the offense has a small increment of additional stockpile, he can change his strategy to a substantially better one.

It is not difficult to propose a much simpler defense strategy which is not quite as good as the van Lint and Pollak one when the offensive stockpile is known, but has essentially equivalent performance if the stockpile is somewhat uncertain (say, to within 10 or 20 percent of some nominal value). One simply allocates defensive missiles proportional to target value, so that the offense cannot obtain any advantage in value destroyed per weapon expended by attacking any subset of the targets. This approach is developed in more detail in Section 5.7.2.

5.6 OFFENSE AND DEFENSE PREALLOCATION STRATEGIES WHEN NEITHER SIDE KNOWS THE OTHER'S ALLOCATION

The preceding three sections have all made the same basic assumption: the offense is able to observe the defensive allocation of missiles to targets before making his own allocation of weapons to targets (offense-last-move). This section, in contrast, examines the much more difficult mathematical problem of determining optimum offense and defense strategies in the (game-theoretic) case when each side knows the other's total stockpile size but not his allocation to targets. Even for the equal-valued target situation of Chapter 4, this model proved quite difficult to analyze for arbitrary values of q_0 and q_1, the survival probabilities of a target attacked by a single unintercepted weapon and a single intercepted weapon, respectively. Therefore, it is not surprising that the results presented in this section are especially fragmentary.

5.6.1 One Offensive Weapon, One Defensive Missile

On page 54 of his book, Dresher (1961) specializes the model as follows: he assumes a group of T targets with values $v_1 > v_2 > \ldots > v_T$, and offensive and defensive stockpiles of size unity ($A = 1$, $D = 1$). He assumes that q_0 is equal to zero, but allows the defensive missile to be unreliable; if ρ is the probability that the defensive missile kills an offensive weapon, then $q_1 = \rho$. The defense strategy consists of specifying the probability with which the single missile is used to defend the ith target, and the offense strategy is defined analogously. Let these strategies be denoted by x_i and y_i, respectively ($\Sigma x_i = 1$, $\Sigma y_i = 1$) and

$$W_j = \sum_{i=1}^{j} 1/v_i \ .$$

Then Dresher shows that the optimum offense strategy is

$$y_i = 1/v_i W_t \ , \qquad i \le t \ ,$$

$$y_i = 0 \ , \qquad i > t \ .$$

The optimum defense strategy is

$$x_i = \frac{1}{\rho}\left(1 - \frac{1}{v_i}\,\frac{t-\rho}{W_t}\right), \qquad i \le t ,$$

$$x_i = 0 , \qquad i > t .$$

Here the quantity t is chosen to maximize the expression $(t-\rho)/W_t$. It is striking that the offense probabilities are inversely proportional to value whenever they are positive.

The expected value of targets destroyed (that is, the value of the game) is given by $(t-\rho)/W_t$ if both sides use optimum strategies.

5.6.2 Piecewise Linear Payoff Functions

On page 124 of his book, Dresher (1961) succeeds in generalizing the solution to arbitrary stockpile sizes of A and D for which $A \ge D$, but finds it necessary to set ρ equal to unity, in effect to permit fractional allocations and also to introduce a somewhat artificial payoff function:

$$V - E(V) = \sum_{i=1}^{T} v_i \max(0, a_i - d_i) .$$

In other words, the damage to a target is proportional to the excess of offensive weapons assigned to it. This is reasonable, for instance, when the target is so extensive that overlapping damage rarely occurs. Interestingly, Dresher finds that the optimum offensive strategy is to attack a single target with the entire stockpile, A; the defense allocates its missiles among the more valuable targets, leaving the less valuable ones undefended.

In a related paper, Cooper and Restrepo (1967) show how the optimum offense and defense strategies can be found when $A \le D$, $\rho = 1$ and the payoff function is

$$\text{Expected gain for the offense} = \sum_{i=1}^{T} K_i(a_i, d_i) ,$$

where $K_i(a_i,d_i) = v_i \cdot (a_i - d_i)$ if $a_i > d_i$ and $K_i(a_i,d_i) = -ha_i$ if $a_i \le d_i$. If $h = 0$, this payoff reduces to Dresher's payoff given above; the second expression reflects a loss to the offense if his weapon allocation to a target is less than or equal to the defensive missile allocation to the same target. Unfortunately, the computations required for the offense and defense strategies are rather extensive;

Cooper and Restrepo work out the optimum strategies only for $(T-1)\,A < D \leq TA$ (T arbitrary) and for $A < D < 2A$ $(T = 3)$.

5.6.3 A Game-Theoretic Solution for Two Targets

The game-theoretic allocation problem for two targets of arbitrary values of v_1 and v_2 has been completely analyzed, if weapons on both sides are perfect, and if each side knows the other's total stockpile (A for the offense, and D for the defense), but not how they are allocated among T targets. Let q_0 be 0 and q_1 be 1 — that is, a target is saved if the defensive allocation is greater than or equal to the offensive allocation, and totally destroyed if the offensive allocation exceeds the defensive allocation. It is more convenient to define the payoff in terms of expected targets lost instead of expected targets saved:

$$
\begin{aligned}
E(V) &= 0 && \text{if} \quad a_1 \leq d_1, \quad a_2 \leq d_2, \\
&= v_2 && \text{if} \quad a_1 \leq d_1, \quad a_2 > d_2, \\
&= v_1 && \text{if} \quad a_1 > d_1, \quad a_2 \leq d_2, \\
&= v_1 + v_2 && \text{if} \quad a_1 > d_1, \quad a_2 > d_2.
\end{aligned}
$$

In specifying the optimum defensive and offensive strategies and the corresponding value of the game (the expected value of targets lost), it is convenient to consider five cases: $D = A - 1$ (neither side dominant), $2A - 1 \geq D \geq A$ (defense dominant), $D \geq 2A$ (defense overwhelming), $D + 2 \leq A \leq 2D + 1$ (offense dominant), and $2D + 2 \leq A$ (offense overwhelming).

When $D = A - 1$, it is easy to obtain the optimum offense and defense strategies. If $v_1 = v_2$, all strategies of both sides are equivalent. If $v_2 \neq v_1$, the unique optimum offense strategy is to allocate all A weapons to the more valuable target; all defense strategies remain equivalent. The value of the game is $V = \max(v_1, v_2)$; the offense destroys the more valuable target.

When $D \geq 2A$, any defense strategy is optimum as long as A or more defensive missiles are allocated to each target. All offense strategies are equivalent, and the value of the game is $V = 0$.

When $2D + 2 \leq A$, any offense strategy is optimum as long as $D + 1$ or more weapons are allocated to each target. All defense strategies are equivalent, and the value of the game is $V = v_1 + v_2$.

In the remaining two cases, the optimum strategies become more complicated. One can determine all optimal strategies, not just one pair of them; but the analysis is tedious. One defines a set of extremal optimum defense strategies, each one of the general form "allocate i missiles to the target of value v_1, and the remaining D - i missiles to the target of value v_2, with probability x_i", where $\Sigma x_i = 1$. To distinguish among different extremal strategies, superscripts can be used on the probabilities x_i. (Extremal optimum offense strategies are defined analogously, with probabilities denoted by y_i.) Any optimum defense strategy can be written as a convex linear combination of extremal optimum defense strategies: if $\Sigma p_j = 1$, then the defense strategy allocating i missiles to the target of value v_i with probability $\sum_j p_j x_i^j$ is optimum. Thus, it is sufficient to specify all extremal optimum strategies corresponding to a given (D,A).

When $2A - 1 \geq D \geq A$, the extremal optimum defense strategies can be characterized as follows. Each extremal optimum defense strategy corresponds to a sequence $M = (m_1, m_2, \ldots, m_K)$ of integers such that $1 \leq m_1 \leq m_2 \leq \ldots \leq m_K \leq R$. It is not difficult to show that there are $\binom{R+K-1}{K}$ such sequences. Here, K is the smallest integer greater than or equal to $(A+1)/(D-A+1)$, and $R = K(D-A+1) - A$. For any sequence M, the corresponding extremal optimum defense strategy is that strategy which allocates $i(D-A+1) - m_i$ missiles to the target with value v_1 (and therefore $D - i(D-A+1) + m_i$ to the target with value v_2) with probability

$$\frac{v_1^{i-1}\, v_2^{K-i}}{v_2^{K-1} + v_1 v_2^{K-2} + \ldots + v_1^{K-2} v_2 + v_1^{K-1}}, \qquad i = 1,2,\ldots,K .$$

The extremal optimum offense strategies are defined analogously. Each one corresponds to a sequence $N = (n_1, n_2, \ldots, n_K)$ of integers such that $(D-A+1) \geq n_1 \geq n_2 \ldots \geq R$. It is not difficult to show that there are $\binom{D-A+K-R+1}{K}$ such sequences. For any sequence N, the corresponding extremal offense strategy is that strategy which allocates $i(D-A+1) - n_i$ weapons to the target with value v_1 (and therefore $A - i(D-A+1) + n_i$ weapons to the target with value v_2) with probability

$$\frac{v_1^{K-i}\, v_2^{i-1}}{v_2^{K-1} + v_1 v_2^{K-2} + \dots + v_1^{K-2} v_2 + v_1^{K-1}}\,, \qquad i = 1,2,\dots,K\ .$$

If both sides use extremal optimum strategies (or convex linear combinations of extremal optimum strategies, which are therefore also optimal), the expected value of targets lost (the value of the game) is

$$V = \frac{v_1 v_2 \left(v_2^{K-2} + v_1 v_2^{K-3} + \dots + v_1^{K-3} v_2 + v_1^{K-2}\right)}{v_2^{K-1} + v_1 v_2^{K-2} + \dots + v_1^{K-2} v_2 + v_1^{K-1}}\,,$$

where K is the smallest integer greater than or equal to $(A+1)/(D-A+1)$.

The following numerical example will illustrate the above formulas. Let $D = 9$ and $A = 6$; then $K = 2$ and $R = 2$. There are three possible (m_1,m_2) sequences: (1,1), (1,2) and (2,2). The corresponding three extremal optimum defense strategies are:

1. Allocate 3 missiles to v_1 with probability $v_2/(v_1+v_2)$,
   Allocate 7 missiles to v_1 with probability $v_1/(v_1+v_2)$.

2. Allocate 3 missiles to v_1 with probability $v_2/(v_1+v_2)$,
   Allocate 6 missiles to v_1 with probability $v_1/(v_1+v_2)$.

3. Allocate 2 missiles to v_1 with probability $v_2/(v_1+v_2)$,
   Allocate 6 missiles to v_1 with probability $v_1/(v_1+v_2)$.

There are six possible (n_1,n_2) sequences: (2,2), (2,3), (2,4), (3,3), (3,4) and (4,4). The corresponding six extremal optimum offense strategies are:

1. Allocate 6 weapons to v_1 with probability $v_1/(v_1+v_2)$,
   Allocate 2 weapons to v_1 with probability $v_2/(v_1+v_2)$.

2. Allocate 6 weapons to v_1 with probability $v_1/(v_1+v_2)$,
   Allocate 1 weapon to v_1 with probability $v_2/(v_1+v_2)$.

3. Allocate 6 weapons to v_1 with probability $v_1/(v_1+v_2)$,
   Allocate 0 weapons to v_1 with probability $v_2/(v_1+v_2)$.

4. Allocate 5 weapons to v_1 with probability $v_1/(v_1+v_2)$,
   Allocate 1 weapon to v_1 with probability $v_2/(v_1+v_2)$.

5. Allocate 5 weapons to v_1 with probability $v_1/(v_1+v_2)$, Allocate 0 weapons to v_1 with probability $v_2/(v_1+v_2)$.

6. Allocate 4 weapons to v_1 with probability $v_1/(v_1+v_2)$, Allocate 0 weapons to v_1 with probability $v_2/(v_1+v_2)$.

The value of the game is $v_1v_2/(v_1+v_2)$.

When $D + 2 \leq A \leq 2D + 1$, the extremal optimum defense and offense strategies can be defined in terms of those already introduced for $2A - 1 \geq D \geq A$. Specifically, set $\overline{D} = A - 2$ and $\overline{A} = D$. The extremal optimum defense strategies are obtained by substituting $\overline{D}$ and $\overline{A}$ into the extremal optimum offense strategy formula, and the extremal optimum offense strategies are obtained by substituting $\overline{D}$ and $\overline{A}$ into the extremal optimum defense strategy formula. The expected value of targets lost is then

$$V = \frac{v_2^L + v_1 v_2^{L-1} + \ldots + v_1^{L-1} v_2 + v_1^L}{v_2^{L-1} + v_1 v_2^{L-2} + \ldots + v_1^{L-2} v_2 + v_1^{L-1}},$$

where L is the smallest integer greater than or equal to $(D+1)/(A-D-1)$.

It should be noted that if one restricts oneself to the simpler problem of finding just one optimal strategy for each side and the value of the game, it should be possible to accomplish somewhat more. Further, if one wishes merely to solve the game for some specific choice of A, D and values of v_i, one can expect to deal with somewhat larger problems (say, five or more targets, depending on the sizes of A and D).

5.6.4 Targets Partitioned Into Homogeneous Classes

What can be done to determine offense and defense strategies for larger numbers of targets of unequal value when $q_0 = 0$ (soft targets) and $q_1 = 1$ (reliable defensive missiles)? Penn (1971) uses the method of Lagrange multipliers (as with Pugh's two-sided offense-last-move strategies discussed in Section 5.3.1) to determine offense and defense preallocation strategies and the expected value of targets lost (the value of the game) for any number of targets of unequal value. Penn's strategies suffer from the same problem of implementation that Matheson's do, namely that, especially for small numbers of targets or weapons, the probability densities called for may not be realizable. Alternatively, Penn's strategies may be considered to have been determined under the constraint that the expected levels of total resource utilization are A and D, respectively. For this reason, his results differ somewhat from the strategies derived for two targets in Section 5.6.3.

Penn shows in his paper that for this modified allocation procedure the Lagrange method does not generate spurious solutions (as it did when applied by Pugh to a two-sided offense-last-move strategy).

Penn's preallocation strategies are a generalization of the Matheson strategies for $q_0 = 0$ and $q_1 = 1$ presented in Section 4.3.2. It will be recalled that Matheson's strategies are useful only when the number of targets is large enough so that the real-valued preallocation strategies can be reasonably approximated by actual targets (for example, if 0.348 of the targets should be defended with 3 missiles apiece, then 2 out of 6 targets are assigned 3 missiles). In order to use Penn's strategies, one must have a large number of targets of each value in order to make missile and weapon assignments approximating the underlying strategies. It should be noted that this problem can be solved conveniently in a specific case by means of linear programming. The technique is the same as that which can be used on the other generalizations of the Matheson problem discussed in Section 4.3.4.

The equations giving the offense and defense strategies and the expected value of targets lost cannot be readily written in terms of the offensive and defensive stockpiles, A and D. Instead, they are given as functions of auxiliary quantities (Lagrange multipliers) λ and ω. In order to find the strategies and payoff, one must first calculate λ and ω by means of a search, given A, D and the target values $v_1, v_2, \ldots, v_n$.

Here a possible inconsistency arises. The formulas relating A and D to λ and ω depend upon whether the offense or the defense is dominant; however, dominance is defined in terms of λ and ω. It is possible that the offense dominant formulas may lead to λ and ω values which specify defense dominance, or vice versa; however, this is unlikely to arise unless one is quite near the boundary of the two regions (in which case using the incorrect strategy won't really matter).

The offense-dominant formulas for determining λ and ω are

$$A = \sum_{j=1}^{n} \left(\frac{3\lambda}{2v_j}\left[v_j/\lambda\right] + 1\right)\left(\left[v_j/\lambda\right] + 1\right),$$

$$D = \sum_{j=1}^{n} \frac{\omega}{2v_j}\left[v_j/\lambda\right]\left(\left[v_j/\lambda\right] + 1\right) .$$

Note that the first equation is a function of λ alone. The corresponding defense-dominant formulas are

$$A = \sum_{j=1}^{n} \frac{\lambda}{2v_j} \left[v_j/\omega\right] \left(\left[v_j/\omega\right] + 1\right) ,$$

$$D = \sum_{j=1}^{n} \left[v_j/\omega\right] \left(1 - \frac{\omega}{2v_j} \left[v_j/\omega\right] - \frac{\omega}{2v_j}\right) .$$

Note that the second equation is a function of ω alone. Once λ and ω have been determined, offense dominance is established if $\omega < \lambda$, and defense dominance if $\omega \geq \lambda$.

Let $x_i(v_j)$ denote the expected fraction of targets of value v_j that are defended by i missiles, and let $y_i(v_j)$ denote the expected fraction of targets of value v_j that are attacked by j weapons. When the offense is dominant, the preallocation strategies are

$$x_i(v_j) = \omega/v_j , \quad i = 1,2,\ldots,\left[v_j/\lambda\right] ,$$

$$x_0(v_j) = 1 - \left[v_j/\lambda\right] (\omega/v_j) ,$$

$$y_i(v_j) = \lambda/v_j , \quad i = 1,2,\ldots,\left[v_j/\lambda\right] ,$$

$$y_{\left[v_j/\lambda\right]+1}(v_j) = 1 - \left[v_j/\lambda\right](\lambda/v_j) .$$

The expected value of targets lost is

$$E(V) = \sum_{j=1}^{n} \left(v_j - \frac{\lambda\omega}{2v_j} \left[v_j/\lambda\right] \left(\left[v_j/\lambda\right] + 1\right)\right) .$$

When the defense is dominant, the preallocation strategies are

$$x_i(v_j) = \omega/v_j , \quad i = 0,1,\ldots,\left[v_j/\omega\right]-1 ,$$

$$x_{\left[v_j/\omega\right]}(v_j) = 1 - \left[v_j/\omega\right] (\omega/v_j) ,$$

$$y_i(v_j) = \lambda/v_j , \quad i = 1,2,\ldots,\left[v_j/\omega\right] ,$$

$$y_0(v_j) = 1 - \left[v_j/\omega\right] (\lambda/v_j) .$$

The expected value of targets lost is

$$E(V) = \sum_{j=1}^{n} \frac{\lambda\omega}{2v_j} \left[v_j/\omega\right] \left(\left[v_j/\omega\right] + 1\right) .$$

These strategies are quite similar to the strategies of Section 4.3.2.

A short example may help clarify this procedure by attempting to apply it to the numerical example of Section 5.6.3. Let $D = 9$ and $A = 6$; assume that one has two targets with values of 1 and 3. Then, using the defense dominant equations, one finds that $\omega = 1/5$ and $\lambda = 6/55$; since $\omega > \lambda$, defense dominance is confirmed.

The offense strategy is

$$y_0(1) = 5/11, \qquad y_i(1) = 6/55 \qquad \text{for} \qquad i = 1,2,\ldots,5 \ ;$$

$$y_0(3) = 5/11, \qquad y_i(3) = 2/55 \qquad \text{for} \qquad i = 1,2,\ldots,15 \ .$$

The defense strategy is

$$x_i(1) = 1/5, \qquad i = 0,1,\ldots,4 \ ;$$

$$x_i(3) = 1/15, \qquad i = 0,1,\ldots,14 \ .$$

The expected value of targets lost is 6/5. Note that these strategies are not implementable and that the expected loss is somewhat different from that derived in Section 5.6.3. This is hardly surprising, since T, A and D are all so small. Goodrich (1970) has extended Penn's methodology to the case of hard targets ($q_0 > 0$).

5.7 DEFENSE STRATEGIES WHEN THE OFFENSIVE STOCKPILE SIZE IS UNKNOWN

In the preceding sections, it was assumed that the defense always knows the stockpile size, A, of the offense (although not how it will be allocated to individual targets). This section suggests ways in which defensive missiles might be allocated if this information is unavailable.

The performance of a defense strategy must be measured against a range of possible attack sizes rather than a single attack size. It seems reasonable to design a defense strategy so that the expected value of targets destroyed is proportional to the (unknown) attack size; if this is so, the offense has no chance of selecting a favorable attack size which will maximize the expected value of targets destroyed per weapon expended. This is an example of a robust strategy as described earlier (Section 1.2). In general, there is no way of finding a strategy which yields a linear return to the offense; however, approximately linear strategies can be found.

5.7.1 Defense Strategy Assuming Offense-Last-Move

Assume that the allocation of defensive missiles to targets of value $v_1, v_2, \ldots, v_T$ can be observed by the offense before an allocation of weapons to targets is made. The object of the defense is to allocate d_i missiles to the ith target so that

$$S_i = \max_{a_i} \left\{ \left(v_i - E\left(i, a_i, d_i\right) \right) / a_i \right\} ;$$

the maximum expected value destroyed per weapon expended is as small as possible. If fractional allocations are permitted, this can be achieved by selecting $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ such that $S_i = K$ at all defended targets $\left(d_i^* > 0\right)$ and $S_i \leq K$ at all undefended targets $\left(d_i^* = 0\right)$. The quantity K is adjusted by trial and error until $\Sigma d_i^* = D$, the defensive stockpile.

It is conjectured that the allocations $\left(d_1^*, d_2^*, \ldots, d_T^*\right)$ and the corresponding $\left(a_1^*, a_2^*, \ldots, a_T^*\right)$ obtained by this method are quite similar to the ones obtained by Pugh (1964) in Section 5.3.1, if one assumes an offensive stockpile $\Sigma a_i^* = A$. If the defense uses the above strategy in place of Pugh's max-min strategy, will it lose more or less expected target value? It might appear that more value will be lost because the information about A is not being utilized; however, since Pugh's method is approximate, the above procedure may lead to results that are better than Pugh's.

5.7.2 Defense Strategy When Neither Side Knows the Other's Allocation

Assume that a set of targets of integral values $v_1, v_2, \ldots, v_T$ is being defended. Assume that each side allocates his missiles or weapons to the targets in ignorance of the other side's allocation. Assume that defensive missiles have perfect reliability — that is, q_1, the probability of an engaged weapon damaging a target, is zero. Finally, assume that an unintercepted weapon damages exactly one unit of target value; that is, it takes v_i unintercepted weapons to destroy totally a target of value v_i. Although this linear damage function is less realistic then an exponential one, the calculation of target damage associated with a specific attack is much simplified. It is conjectured that the approximate linearity of the defensive strategies will not be much affected by using a more realistic damage function.

One way to derive a defense strategy is to construct one analogous to a strategy derived for equal-valued targets. In Section 4.3.2 it was stated that if $q_0 = 0$ and $q_1 = 1$, and if d defensive missiles per target are available, then if d is an integer, the optimum defensive allocation assigns $0,1,2,\ldots,2d-1$ or $2d$ missiles to the defense of a target, each with probability $1/(2d+1)$. If one has a group of targets with different values, the analogous strategy is obvious: If v_i is an integer, assign $0,1,2,\ldots,2v_i-1$ or $2v_i$ defenseive missiles to a target of value v_i, each with probability $1/(2v_i+1)$. Note that this implies that one has a defensive stockpile of size $\Sigma v_i = D$. For stockpile sizes $D = K\Sigma v_i$, where K is an integer, one can scale up the allocations proportionately: $0,1,2,\ldots,2Kv_i-1$ or $2Kv_i$ defensive missiles assigned with probabilities $1/(2Kv_i+1)$, or $0,K,2K,\ldots,K(2v_i-1)$ or $2Kv_i$ defensive missiles assigned with probabilities $1/(2v_i+1)$. For intermediate values of D, various approximations to the allocations can be devised.

The following simple example shows how such a defense strategy can be simultaneously realized for all targets. Assume that one has three targets of values $v_1 = 2$, $v_2 = v_3 = 1$ and a stockpile D of four defensive missiles. The 15 possible allocations of missiles to targets (v_1,v_2,v_3) are listed below.

Probability	Allocation	Probability	Allocation
p_1	(4,0,0)	p_9	(0,2,2)
p_2	(3,1,0)	p_{10}	(0,4,0)
p_3	(3,0,1)	p_{11}	(0,0,4)
p_4	(2,1,1)	p_{12}	(0,3,1)
p_5	(2,0,2)	p_{13}	(0,0,3)
p_6	(2,2,0)	p_{14}	(1,3,0)
p_7	(1,2,1)	p_{15}	(1,0,3)
p_8	(1,1,2)		

How should one choose the p_i so that $p_i \geq 0$, $\Sigma p_i = 1$, and so that the allocation of missiles to individual targets have the pattern specified above? To begin with, it is clear that $p_{10} = p_{11} = \ldots = p_{15} = 0$, for these allocate three or more missiles to targets having a value of one, and this is more than the maximum allowed. For the target of value v_1, the p_i must satisfy the following linear equations:

$$p_1 = 1/5 \quad ,$$

$$p_2 + p_3 = 1/5 \quad ,$$

$$p_4 + p_5 + p_6 = 1/5 \quad ,$$

$$p_7 + p_8 = 1/5 \quad ,$$

$$p_9 = 1/5 \quad .$$

For the target of value v_2, the p_i must satisfy the following linear equations:

$$p_1 + p_3 + p_5 = 1/3 \quad ,$$

$$p_2 + p_4 + p_8 = 1/3 \quad ,$$

$$p_6 + p_7 + p_9 = 1/3 \quad .$$

Finally, for the target of value v_3, the p_i must satisfy the following linear equations:

$$p_1 + p_2 + p_6 = 1/3 \quad ,$$

$$p_3 + p_4 + p_7 = 1/3 \quad ,$$

$$p_5 + p_8 + p_9 = 1/3 \quad .$$

This constrained set of linear equations can be solved by Gaussian elimination to obtain

$$\begin{aligned} p_1 &= 1/5 \quad , & p_6 &= 1/15 - p_5 \quad , \\ p_2 &= 1/15 + p_5 \quad , & p_7 &= 1/15 + p_5 \quad , \\ p_3 &= 2/15 - p_5 \quad , & p_8 &= 2/15 - p_5 \quad , \\ p_4 &= 2/15 \quad , & p_9 &= 1/5 \quad , \end{aligned}$$

where p_5 can take on any value in the range $0 \leq p_5 \leq 1/15$. In other words, there are an infinite number of possible solutions to the problem. In other problems, however, it may happen that no solution is possible.

The performance of this defensive strategy cannot be assessed until one specifies the offensive strategy for each possible stockpile value A. Assume that the offense knows that the defense is using this strategy; he will then allocate his weapons to targets in such a way as to maximize the expected value of target destruction. By considering all possible offensive allocations, it is not difficult to discover the optimum allocation and the corresponding payoff for each value of A:

Stockpile Size	Allocation of Weapons to Targets	Expected Value Destroyed
1	(0,0,1),(0,1,0)	5/15
2	(0,1,1)	10/15
3	(0,1,2),(0,2,1),(3,0,0)	15/15
4	(4,0,0)	21/15
5	(5,0,0)	27/15
6	(5,0,1)	32/15
7	(5,1,1)	37/15
8	(5,1,2),(5,2,1)	42/15
9	(5,2,2)	47/15
10	(5,2,3),(5,3,2)	52/15
11	(5,3,3)	57/15
12	(6,3,3)	60/15

It is evident that the expected value destroyed is quite close to a linear function of attack size A.

If the value of one target exceeds half the number of defense missiles available, the defensive strategy must be modified slightly. This occurs, for example, when $v_1 = 3$, $v_2 = v_3 = 1$, and $D = 5$. In this situation, the valuable target is defended with $2v_i - D$, $2v_i - D+1, \ldots, D-1$ or D missiles, each with probability $1/(2D-2v_i+1)$. The p_i can be found by Gaussian elimination as before.

The reader should remember that no optimum defense strategies have been derived in this section; instead, a plausible defense strategy has been proposed. It is entirely possible that an optimum strategy would yield a curve of expected value of targets destroyed versus offensive stockpile size that is below the curve for the plausible strategy. However, it is conjectured that the plausible defense strategy performs nearly as well as the optimum one. It is clearly superior to a defense strategy which allocates defensive missiles proportional to target value (a simple defense strategy useful when the offense has the last move, as discussed in the last paragraph of Section 5.5.2). In fact, one can derive the following result. Let G denote the ratio of the number of offensive weapons required to obtain a given expected destruction when the defense uses the randomized allocation, to the number of offensive weapons required to obtain the same expected destruction when the defense uses a fixed allocation proportional to target value. Then, if $D = \Sigma v_i$,

$$G = \frac{1}{2}\sum_{i=1}^{T}\left(\min(D, 2v_i) + v_i\right)\Big/\sum_{i=1}^{T} v_i \; .$$

The quantity G is bounded by 1 and 1.5. In the example given earlier in this section ($v_1 = 2$, $v_2 = v_3 = 1$), $G = 1.5$; in general, $G = 1.5$ if $2 \max(v_i) \leq \Sigma v_i$.

5.8 ATTACKER-ORIENTED DEFENSE STRATEGIES

So far, this chapter has been entirely concerned with preallocation defense strategies — those in which the defense assigns missiles to the defense of specific targets. In Chapter 4 it was pointed out that it sometimes may be impossible for the defense to know which targets weapons are directed at; in such a situation, the defense must use an attacker-oriented defense instead. It was shown there that under certain circumstances an attacker-oriented defense may actually lead to a larger expected fraction of targets saved than a preallocation defense, and therefore would be preferred even if the defense knows the targets weapons are directed at.

If one is restricted to offense-last-move, the uniform attacker-oriented defense strategy described in Section 4.6.2 is optimum when the targets have different values and the objective is to maximize the expected value of targets saved. The proof is identical to the one given in that section. Furthermore, the algorithm presented there for the determination of the optimum offensive strategy against the uniform defense strategy can also be carried through, but it is not known whether the algorithm is reliable in this case. Burr and Graham (1970) conjecture that the algorithm will find the optimum, providing that no degeneracies occur. For details of the algorithm, the reader is referred to Section 4.6.2.

What if neither side knows the strategy employed by the other? Assume that one is defending T point targets having values $v_1 \geq v_2 \geq \ldots \geq v_T$ with a stockpile of D missiles, each of which has a reliability of ρ. Assume that the offense has a stockpile of A weapons; if a weapon is not destroyed by a missile, it will destroy the target it is directed against with probability p. The quantities A, D, T, ρ, p and the v_i are known to both the offense and the defense. The defense is unable to determine which target a weapon is directed against.

As in Section 4.6.1, the optimum defense in this case is a uniform random attacker-oriented strategy; the proof is the same as that of Section 4.6.1. That is, he allocates $[D/A]$ missiles randomly to $A - D + A[D/A]$ incoming weapons, and $[D/A] + 1$ missiles to the remaining $D - A[D/A]$ incoming weapons. The offense has no way of knowing which number of missiles have actually been allocated to a given weapon, unless D/A is an integer. When it is an integer, the problem becomes essentially that of Section 3.2.4, and can be solved exactly by the methods there. Moreover, the approximate solution given below becomes exact.

If D/A is not an integer, determining the optimum offense strategy is considerably more complicated. The strategy can be approximated very closely as follows. First, decide which targets are worth attacking. The offense allocates weapons to the T_0 targets of greatest value, where T_0 is the maximum value of i satisfying the inequality

$$v_i \geqq \left(\prod_{j=1}^{i} v_j \right)^{1/(i-1)} Q^{(A-1)/(i-1)} , \qquad 1 \leq i \leq T ,$$

and Q is the probability that a weapon will not destroy the target it is directed against:

$$Q = 1 - p\left(1 - p_0\right) ,$$

where

$$p_0 = \left(1 - D/A + [D/A]\right)\left(1 - (1-\rho)^{[D/A]}\right) + \left(D/A - [D/A]\right)\left(1 - (1-\rho)^{[D/A]+1}\right) .$$

The reason that the strategy given here is only an approximation is the fact that the probability that a target will survive an attack of k weapons is not Q^k, since the events are not independent. However, if A is reasonably large, this discrepancy should be negligible.

Granting this approximation, the problem becomes essentially identical to that of Section 3.2.4, and which was also mentioned in Section 5.1.0. The number of weapons assigned to target v_j, $1 \leq j \leq T_0$, is given by

$$a_j = \left(\log C - \log v_j\right)/\log Q ,$$

where

$$C = \left(\prod_{i=1}^{T_0} v_i\right)^{1/T_0} Q^{A/T_0} .$$

This leads to nonintegral a_j; however, they can be rounded off to yield an exact solution to the problem (except for the above-mentioned approximation concerning Q^k). The fact that such rounding is possible can be deduced from results of Section 3.2.4, and also follows directly from the result of Dym and Schwartz (1969) mentioned in Section 5.1.

If both sides use these (unrounded) strategies, the expected value of targets destroyed is closely approximated by

$$E(V) = \sum_{j=1}^{T_0} v_j - \left(\prod_{j=1}^{T_0} v_j\right)^{1/T_0} T_0 Q^{A/T_0} .$$

If the defense is able to determine which target each weapon is directed against, he has the option of using a preallocation strategy instead. It is important to compare the above E(V) with the corresponding preallocation E(V) to see which strategy is preferable for the defense.

5.9 SUMMARY

This chapter extends many of the concepts introduced in the previous chapter to the situation in which targets have unequal values; the criterion then generalizes to expected value saved. Not surprisingly, general strategies are much more difficult to obtain; even the problem of allocating weapons to targets in the no-defense case in order to minimize the expected value saved is not trivial.

Either dynamic programming or Lagrange multipliers can solve the one-sided problem of allocating weapons against a known defense (or missiles against a known offense); however, the two-sided problem of the optimum defense allocation of missiles, given that the offense can observe this and then allocate weapons, is quite difficult to solve. Several models are presented, differing principally in the degree of generality of the payoff function (the expected value saved when i missiles intercept j weapons at a single target).

When neither side knows the other's preallocation before making his own, limited results are known. An exact allocation for two targets of different values (or for a single weapon against a single missile) can be given. One can also give an approximate allocation (not stockpile-constrained) for more realistic numbers of targets, weapons and missiles if missiles and weapons are both perfectly reliable.

The chapter concludes with a brief analysis of defense strategies required when the defense cannot determine which target a weapon is directed against when an intercept must be decided upon. It is suggested that a defense allocation that makes expected damage proportional to attack size is prudent if (contrary to the assumption throughout the rest of this chapter) the attack size is unknown.

CHAPTER SIX

APPLICATIONS OF OFFENSE AND DEFENSE STRATEGIES TO SPECIAL PROBLEMS

This final chapter examines some problems which arise when the idealized offense and defense strategies of the preceding chapters are applied to various special situations. Unfortunately, there is very little material in the mathematical literature dealing with these applications. The mathematical models are in general quite complicated; it is difficult to obtain analytic results except in very special circumstances. In many cases, it is necessary to resort to a high-speed digital computer either to search through many alternative strategies, or to simulate the offense-defense problem using Monte Carlo techniques. Indeed, it happens all too frequently that none of the above approaches gives very satisfactory answers to the mathematical problem.

This chapter presents a more or less systematic classification of each problem according to its input assumptions. It is hoped that the framework is sufficiently broad so that a reader with a specific problem can identify its mathematical model and contrast it with related models. In particular, he can decide whether or not it is appropriate to approximate his model by a mathematically more tractable one.

6.1 ATTACKS ON THE DEFENSE SYSTEM

The primary objective of the offense is to minimize the number of targets surviving. When the targets are defended, he has two ways of accomplishing this: (1) attack the targets directly; (2) attack the defense system first and then the targets. Because undefended targets are ordinarily much more vulnerable to destruction than defended ones, the offense may find the second option attractive even though a part of his stockpile must first be allocated to the defense system attack.

Usually, there is one component of the defense system which is the most profitable to attack, either because of its inherent vulnerability to damage, or because of the relatively small numbers of this component deployed. Radars, for example, tend to be difficult to protect against blast damage because of their large size, and because under some circumstances they can be rendered temporarily useless by atmospheric blackout from an otherwise harmless burst of a weapon. Furthermore, radars are ordinarily quite expensive and therefore few are used in a defense system. In this section, attacks against the defense system are described in terms of attacks against its radars; however, the reader should understand that in some applications it may be more profitable to attack the control centers, the communications or the tactical computer instead.

In general, the mathematical model of Chapter 4 is assumed to hold. Specifically, one has a set of T identical point targets to be defended, and a set of R identical radars to carry out this task. It is assumed that any one of the radars can successfully defend the targets; the offense must destroy all R radars before he can attack an undefended set of targets. The targets and the radars are located sufficiently far apart so that an offensive weapon which destroys one target or radar does not affect any other target or radar. The defense can carry out attack evaluation (that is, he knows which target or radar is being attacked in time to make an intercept if desired), but he cannot do damage assessment (change his strategy in the course of the engagement depending upon which targets or radars have been destroyed). The latter assumption is rather unrealistic, as the defense should know immediately if one of his radars is destroyed; however, the defense in many cases will be defending only one radar anyway so no reallocation is possible. In any case, the assumption provides a bound on the outcome.

The defense has a stockpile of D missiles, and the offense a stockpile of A weapons; each side knows the value of both D and A. The probability that an unintercepted (or intercepted) weapon does not destroy the target at which it is aimed is q_0 (or q_1); the analogous probability of radar survival is q_{0r} (or q_{1r}). These quantities are known by both the defense and the offense. Although it is appropriate to restrict the defense to one-on-one missile engagements when defending targets, it is not necessarily reasonable to restrict the defense to one-on-one missile engagements when defending radars. In particular, one can assume three different defensive strategies:

1. $q_{1r} = q_{0r} + \rho\left(1-q_{0r}\right)$ (one-on-one) ,

2. $q_{1r} = q_{0r} + \left(1 - (1-\rho)^2\right)\left(1-q_{0r}\right)$ (two-on-one) ,

3. $q_{1r} = q_{0r} + \left(1 - (1-\rho)^2\right)\left(1-q_{0r}\right)$ (shoot-look-shoot) ,

where ρ is the defensive missile kill probability. The second and third strategies give identical q_{1r}, but the second strategy uses two defensive missiles whereas the third strategy uses one missile with probability ρ and two with probability $(1-\rho)$. When ρ is equal to unity, only the first defensive strategy need be used.

The criterion of effectiveness is E(f), the expected fraction of targets saved in the set. It has the generic form

$$E(f) = PE_u(f) + (1-P)\, E_d(f) \quad ,$$

where P is the probability that all radars are destroyed, $E_u(f)$ is the expected fraction of targets saved if undefended, and $E_d(f)$ is the expected fraction of targets saved if defended. Note that $E_d(f)$ can be a function of P. The object of the offense is to minimize this quantity (subject to his stockpile limitation), and the corresponding goal of the defense is to maximize this quantity. It should be recognized that the expected fraction of targets saved may be a misleading criterion. In the target attacks discussed in Chapter 4, the fraction of targets saved was a random variable having a probability density function with a single mode centered on E(f). However, the fraction of targets saved in a radar attack has a bimodal probability density function; most of the probability content in this density is located in the vicinity of $E_u(f)$ and $E_d(f)$, not at E(f). On the other hand, if the offense independently attacks many sets or modules of radar-defended targets, the bimodal character tends to vanish -- one is simply summing a number of binomial variables, which leads to a Gaussian distribution of expected fraction saved. Nevertheless, after carrying out an optimization based in E(f), it is prudent to record the probability that all radars are destroyed as well as the two dependent expectations $E_u(f)$ and $E_d(f)$.

If a_i represents the number of attackers and d_i the number of defenders for the ith target,

$$E_d(f) = \frac{1}{T}\sum_{i=1}^{T} q_1^{\min(a_i,d_i)}\, q_0^{\max(0,a_i-d_i)} ,$$

$$E_u(f) = \frac{1}{T}\sum_{i=1}^{T} q_0^{a_i} .$$

The probability of radar destruction, P, takes on a variety of functional forms depending on the defense strategies used. For example, if the ith radar is attacked by a_{ir} weapons and defended by d_{ir} missiles, and engagements are one-on-one, then

$$P = \prod_{i=1}^{R} \left(1 - q_{1r}^{\min(a_{ir},d_{ir})}\, q_{0r}^{\max(0,a_{ir}-d_{ir})}\right) .$$

6.1.1 Some Simple Models Involving Reliable Missiles and Soft Radars

The best way to illustrate the complexities of radar attack offense and defense strategies is to start with a very simple problem, and add on complicating factors one at a time. Assume that the offense has the last move; that is, it can make its allocation after observing how the defense has allocated missiles to the defense of radars and targets. This situation will occur, for example, if the defensive missiles have a limited range and the targets and radars are far apart; each defended point has its own stockpile. Assume that defensive missiles are reliable (that is, $q_1 = q_{1r} = 1$); assume also that radars are completely vulnerable to attack (that is, $q_{0r} = 0$). Clearly, $d_1 = \ldots = d_T$ and $d_{1r} = \ldots = d_{Kr}$. However, the balance between Σd_{ir} and Σd_i depends upon the strength of the offense. If the offense is so strong that he can afford to allocate his weapons evenly among the defended targets (instead of attacking a subset of them), then Σd_{ir} should be set equal to Σd_i. However, not all offenses are this strong; for example, let $D = 48$, $A = 36$, $T = 12$ and $q_0 = 2/3$, and suppose that the defense allocates 24 missiles to radar defense and two missiles apiece to each target. If the offense first neutralizes the radar, he has one weapon left per undefended target, and E(f), the expected fraction saved, is 2/3. However, if he ignores the radars and attacks four targets with four weapons apiece and four targets with five weapons apiece, then E(f) is only $(1/3) + (1/3)(2/3)^3 + (1/3)(2/3)^2 = 47/81$, and the offense will prefer this attack. In other words, it may be worthwhile for the defense to shift some missiles from radars to targets in order to make E(f) equal for the two attacks. Note that the defense needs only one radar; additional radars contribute nothing to the defense in this simple model. This model is discussed by Shapiro, Abramson and Coburn (1966).

Suppose now that the defense can place his missiles in a central stockpile, and use any missile to defend any radar or target in the set. The offense no longer has the last move; for simplicity assume instead that the defense has the last move. As usual, the offense has two options — attack the targets ignoring the radars, or attack the radars first and the targets subsequently.

How does the defense respond to an attack? The defense must defend one radar against attack (selected at random so the offense does not know which) as long as defensive missiles remain in the stockpile; if he allows a single weapon to penetrate, the radar is destroyed and the remaining stockpile is wasted. If the defense still has missiles available when the attack against targets begins, he matches missiles to weapons starting with the most lightly-attacked targets; therefore the attacker will attack targets as evenly as possible.

If the offense elects to attack radars first, he will attack all radars with the same number of weapons; he knows that the defense need only allocate missiles to the defense of the most lightly-attacked radar. Note that he attacks the radars in order to reduce or exhaust the defensive stockpile, not to destroy the radars themselves; it is impossible for him to destroy all radars until no defensive missiles remain. However, it is instructive to examine the form of the attack even in this drastically simplified and less-than-realistic situation. This is perhaps the simplest model in which the following question can be meaningfully asked: how many radars does the defense require in order that the payoff (the expected number of targets saved) in a radar-first attack is less than or equal to the payoff for a targets-only attack?

Rather than write down general formulas, it is useful to exhibit the calculations carried out for a simple example. Let $D = 12$, $A = 36$ and $q_0 = 2/3$; assume that there are $T = 12$ targets and $R = 1$ radar (with defense-last-move, all targets will be saved unless $A > D$). Suppose that the offense attacks the radar with 6 weapons; the defense responds with 6 missiles. Finally the offense attacks 6 targets with 2 weapons each and 6 targets with 3 weapons each, and the defense responds by matching the offense at 3 of the 2-weapon targets:

$$E(f) = (1/4) + (1/4)(2/3)^2 + (1/2)(2/3)^3 = 0.509 \ .$$

Carrying out similar calculations for all possible radar attacks from 0 to 12, are finds that the preferred attack is 12 weapons, exhausting the defense and yielding an E(f) of only $(2/3)^2 = 0.444$. Clearly, one radar is insufficient; in fact this can be demonstrated by the formula

$$\frac{D}{A} + \left(1 - \frac{D}{A}\right) q_0^{A/T} \geq q_0^{(A-D)/T} \ .$$

Suppose now that there are $R = 2$ radars. Now the defense has a two-to-one advantage in trading missiles for weapons during the radar attack. A sample of possible attacks is given in the table below:

Size of Attack on Each Radar	Weapons for Target Attack	Missiles for Target Defense	E(f)
0	36	12	0.531
3	30	9	0.574
6	24	6	0.583
9	18	3	0.637
12	12	0	0.667

The radars are no longer the soft spot in the defense; the attacker can achieve his best result by attacking targets directly.

The offense can decide upon his allocation before the attack takes place. However, in the defense-last move model given above, the defense must make decisions during the course of the attack. In particular, the defense must decide whether a radar or target attack is taking place and instruct his missiles accordingly; furthermore, when the radar attack is finished, the defense must immediately work out the appropriate strategy corresponding to the remaining offensive and defensive stockpiles. All this demands a great deal of up-to-date information on the part of the defense, and it may be prudent to construct a defense strategy which does not depend upon this capability. Specifically, one can in principle design a defense strategy analogous to the Matheson strategy of Section 4.3.1; that is, the defense decides before the attack to defend a fraction R_1 of the radars with one missile each, R_2 of the radars with two missiles each, and so on; the actual radars are selected at random. The remaining defense missiles are analogously allocated to the targets. Using this strategy, the defense may end up with radar-allocated missiles which are never used to defend targets. It is conjectured that this defense strategy is considerably inferior to the defense-last-move strategy; undoubtedly more radars would be required to reduce the payoff of a radar attack below that for a target attack. Such a strategy appears to be quite difficult to derive in general.

6.1.2 Radars are Resistant to Damage

Suppose now that the model is generalized to allow for radars resistant to damage; that is, suppose that $q_0 > q_{0r} > 0$. If the offense has the last move, it is clear that the optimum defensive allocation divides the missiles assigned to targets equally among the targets. On the other hand, it does not matter how the missiles assigned to radar defense are allocated, because all the radars must be destroyed to nullify the defense. In short, the defense strategy can be characterized by a single quantity D_r, the number of missiles (out of a stockpile of size D) assigned to radar defense.

Unfortunately, the offense strategy cannot be specified so simply. Not only must he determine A_r (the number of weapons assigned to the radars, out of a stockpile of size A), but also he must decide upon the allocation to targets not knowing whether they can be defended or not by their missile stockpile. All that the offense can do is to control the <u>probability</u> that the targets will be undefended (i.e., that the radars will be destroyed) by the number of excess $A_r > D_r$ he assigns to the radars to exhaust the radar defense and then destroy the radars themselves. Clearly, the offensive allocations against defended or undefended targets will be somewhat different, and the best the offense can do is a compromise allocation based on the probability of radar kill.

The form of the strategy is now clear. For any choice of D_r by the defense, the offense will pick that A_r and target allocation which minimizes E(f), the expected fraction of targets saved; on the other hand, the defense will pick that D_r which maximizes this minimizing E(f). Unfortunately, it is very tedious to carry out these calculations in practice, except for extremely small numbers of weapons, missiles and targets.

If the defense can place his missiles in a central stockpile, denying the offense the last move, the problem becomes significantly more involved. As in the last section, it seems plausible to assume that if the defense has the last move he will select one radar at random to defend. However, the optimality of this strategy remains to be demonstrated. More generally, it would be of interest to generalize the Matheson preallocation strategies to the case of radar defense; however, this appears very difficult to carry out.

6.1.3 Unreliable Defensive Missiles

Finally, what is the form of the optimum defense strategy (and offense strategy) when the defensive missile reliability, ρ, is less than unity? Unfortunately, few theoretical models for finding such strategies have been proposed. Most studies assume a relatively limited set of strategies and find the best one by examining all possibilities. The mathematical models of Chapter 4 do not seem particularly useful for deriving optimum defense strategies for the radars, because the objectives are different: target defense strategies are designed to minimize the expected fraction of targets destroyed, whereas radar defense strategies are designed to maximize the probability that at least one radar survives.

Any comprehensive radar defense strategy must take the following considerations into account:

1. How many radars should be defended? The advantage of defending a subset of radars is that the offense is likely to waste weapons overkilling undefended radars.

2. Should the defense use preallocation strategies (assign a fixed number of missiles to each radar before the attack, concealing this from the offense) or group preferential strategies (defend a random subset of radars against any attackers as long as any missiles remain in the stockpile)? If an unintercepted offensive weapon has a high probability of destroying the last surviving radar, it seems unwise to hold back any missiles for target defense.

3. Under what circumstances should the defense switch from a one-on-one defense to a two-on-one defense? For example, a two-on-one defense of a single radar may be a much better strategy than a one-on-one defense of two radars. Should two-on-one be used early in the engagement, and one-on-one as the missile stockpile is depleted? Of course, if shoot-look-shoot is available, it should be used.

4. The defense should know when each radar is destroyed. Can a damage assessment defense be designed to take advantage of this ability?

The target defense strategy is somewhat simpler to define; typically, one can use the Matheson strategy of Chapter 4, or a defense-last-move strategy as an upper bound on the expected fraction of targets saved. Note that the defense will not know the number of offensive weapons allocated to targets until the target attack begins; ote also that if a shoot-look-shoot radar defense or a group radar defense is used, the defense will not even know the number of its own missiles available for target defense until the radar attack is done. Can one devise an overall defense strategy which does not require decisions like these during the course of the engagement? Is such a strategy significantly inferior in terms of the expected fraction of surviving targets? Turning to the offense, can he observe whether or not his radar attack has succeeded and then select his target strategy, or must he decide on this prior to the attack?

Assume that the offense allocates A_r weapons to the radar attack and $A - A_r$ to the target attack. The defense can observe A_r before deciding how many missiles to allocate to radar defense; therefore, the offense selects A_r and the defense selects D_r to satisfy

$$E(f) = \max_{D_r} \min_{A_r} \left(PE_u(f) + (1-P)\, E_d(f) \right) .$$

The offense allocates A_r weapons evenly to radars; the defense allocates missiles evenly to radars (either one-on-one or two-on-one) until D_r is exhausted. Shoot-look-shoot is not allowed. The defense then allocates the remaining $D - D_r$ missiles to targets according to a Matheson strategy; the offense allocates the remaining $A - A_r$ weapons to targets according to a Matheson strategy if he observes that one or more radars survive, or uniformly if he observes that all radars have been destroyed. (If the defense cannot determine $A - A_r$, or the offense cannot assess radar damage, one can derive lower bounds for the expected fraction of surviving targets $E(f)$.) Finally, for the one-radar case, one can calculate $E(f)$ using a shoot-look-shoot strategy. Assuming $D > A_r$, each weapon attacking the radar is allocated one missile, and a second missile if the first one fails; any missiles left over from the radar defense are then allocated to target defense using a Matheson strategy. Note that no optimization of either A_r or D_r is attempted in this more complex model.

It is possible that the offense will want to reduce the uncertainty of the radar attack outcome by designing his attack so as to have a high assurance that all radars will be destroyed: $P \geq P_0$. The outcome of the attack can then be characterized by two parameters instead of three:

$E_u(f)$, the expected fraction saved if the targets are undefended, and

P_0, the probability that the attack fails to achieve this goal.

Unfortunately, the defense is not likely to know the value of P_0 selected by the offense; the defense must therefore evaluate the sensitivity of his strategy to a variety of different possible P_0. This of course, negates one of the most attractive characteristics of this model: if P_0 is known, the number of alternative attacks to be evaluated is drastically reduced (either $P = 0$ or $P = P_0$).

Suppose that the defense is restricted to a one-on-one defense of both radars and targets. Assume that the defensive missile reliability is ρ, and that q_0 and q_{0r} are the survival probabilities of targets and radars, respectively, against unintercepted weapons directed at them. Assume that the offense has a stockpile A and the defense a stockpile D. Assume that the defense intercepts each attacker as long as his stockpile holds out. Assume that the offense attacks the R radars with a_r weapons apiece and the T targets with a_t weapons apiece:

$$A = Ra_r + Ta_t \; .$$

The quantity a_r is chosen by the offense to minimize the expected fraction of targets saved. How many radars must the defense supply in order to make it more attractive for the offense to attack targets with his entire force? In other words, how many radars are needed to keep them from being the soft spot in the defense?

A digital computer can be used to find the required number of radars in case $D \geq A$; one can prove that in this case the number is independent of D, so that one may assume $D = A$. As D drops below A, so does the required number of radars; but this will not be considered. The expected fraction of targets saved in a target-only attack is approximated by

$$E_t(f) = \left(q_0 + \rho\left(1-q_0\right)\right)^{A/T} \; .$$

The expected fraction of targets saved in a mixed target-radar attack is approximated by

$$E_m(f) = Kq_0^{\left(A-Ra_r\right)/T} + (1-K)\left(q_0 + \rho\left(1-q_0\right)\right)^{\left(A-Ra_r\right)/T} \; ,$$

where K, the probability that all R radars are destroyed, is given by

$$K = \left(1 - \left(q_{0r} + \rho\left(1-q_{0r}\right)\right)^{a_r}\right)^R .$$

Note that these equations are strictly correct only when all exponents are integers; more complicated formulae for non-integral exponents can be used instead if greater accuracy is desired.

How does one determine the minimum necessary R? In principle, one selects a value of R and then computes $E_m(f)$ as a function of a_r over the range $0 \le a_r \le A/R$; the smallest integral R for which $E_m(f) > E_t(f)$ for <u>all</u> a_r in this range is the required number of radars. Although this is very difficult to obtain analytically, a digital computer can be easily programmed to start with R equal to one and increase it until $E_m(f) > E_t(f)$ always.

Consider an example in which $T = 20$, $\rho = 0.8$, $q_0 = 0.8$, $q_{0r} = 0.6$ and $A = D = 300$. It turns out that the offense should attack six radars with 30 weapons each and the twenty targets with 6 weapons each. This attack destroys all six radars with probability 0.60, and leads to an $E_m(f)$ of 0.47, which is less than $E_t(f) = 0.54$, the target-only attack of 15 weapons on each target. However, if the defense provides a seventh radar the offense will do better to attack the targets.

This model can be readily modified to incorporate P_0, the minimum acceptable probability of radar destruction in a mixed target-radar attack. However, it is worth remembering that this model allows for a rather restrictive defense strategy — it is possible that the defense could get by with fewer radars if he were allowed to defend targets and radars other than uniformly. However, it is difficult for the defense to specify preallocation models unless he knows how many weapons the offense plans to allocate to the radars.

6.1.4 <u>A Model With Offensive Damage Assessment</u>

Brodheim, Herzer and Russ (1967) discuss attacks on the defense system under a somewhat different set of assumptions than the ones so far considered in this section. As before, it is assumed that one has a set of T identical point targets to be defended, and a set of R identical radars to carry out this task. Any one of the radars can successfully defend all of the targets; the attacker must destroy all R radars before he can attack an undefended set of targets. The targets and the radars are located sufficiently far apart so that an offensive weapon which destroys one target or radar does not affect any other target or radar. The defense has a stockpile of D missiles, and this quantity is known to the offense. The probability that an unintercepted weapon destroys the target it is aimed at is p, and the corresponding probability for the radar is p_r. The probability that a missile successfully intercepts a weapon is equal to ρ. Both the offense and the defense know ρ, p and p_r.

However, there are several important differences in the assumptions. The defense no longer knows the size of the offensive weapon stockpile, but he does know that the offense plans to continue attacking until all T targets have been destroyed. The offense attacks sequentially, one weapon at a time being assigned either to a radar or to a target; the defense knows that the object of the offense is to keep attacking until he has destroyed all T targets. Furthermore, the offense can carry out perfect damage assessment between firings; he knows exactly which targets and radars have been destroyed. Of all these assumptions, the offensive knowledge of target damage seems to be the most difficult to achieve in practice.

The criterion of effectiveness is the expected number of offensive weapons required to destroy T targets; the defense seeks to maximize this by assigning a suitable number of defensive missiles to each offensive weapon (this number depends upon the remaining defensive stockpile size, as well as whether a target or radar is being defended), and the offense seeks to minimize this by deciding whether to attack a target or a radar with the next weapon (this depends upon the number of surviving targets and radars, and the remaining defensive missile stockpile).

It is impossible to write down in analytic form the offense and defense strategies and the expected number of weapons needed to destroy T targets. However, one can derive these quantities recursively (by the method of dynamic programming), working back from the end of the engagement. Let $f_t(i,j,k)$ be defined as the expected number of offensive weapons required to destroy i surviving targets, given j surviving radars and k remaining defensive missiles, and given that the offense next attacks a target. Let $f_r(i,j,k)$ be the analogous expected number of weapons, given that the offense next attacks a radar. Let

$$f(i,j,k) = \min\left(f_t(i,j,k),\ f_r(i,j,k)\right) .$$

The initial conditions can be readily calculated:

$$f_t(0,j,k) = f_r(0,j,k) = 0 \qquad \text{for all j and k} ,$$

$$f_t(i,j,0) = f_t(i,k,0) = i/p \qquad \text{for all i, j and k} ,$$

$$f_r(i,j,0) = f_r(i,0,k) = 1 + i/p \qquad \text{for all i, j and k} .$$

The recursive equations are

$$f_t(i,j,k) = \max_{0\le m\le k} \left(1 + f(i,j,k-m)\left(1 - p(1-\rho)^m\right) + f(i-1,j,k-m)p(1-\rho)^m\right),$$

$$f_r(i,j,k) = \max_{0\le m\le k} \left(1 + f(i,j,k-m)\left(1 - p_r(1-\rho)^m\right) + f(i,j-1,k-m)p_r(1-\rho)^m\right),$$

$$f(i,j,k) = \min\left(f_t(i,j,k),\ f_r(i,j,k)\right).$$

Using these, one can calculate the expected number of weapons required, as well as the associated ffense and defense strategies, for the case $i = T$, $j = R$ and $k = D$. For most cases of practical interest a digital computer will be needed. Note that the above also solves the problem if the attacker only wishes to destroy $I < T$ targets; in fact this is the problem Brodheim, Herzer and Russ actually considered. In fact, the target structure really plays no part in the problem.

Brodheim, Herzer and Russ conjecture that the optimum offense strategy always takes one of two forms — all weapons are assigned to targets, or else weapons are assigned to radars until all radars are destroyed. It would be of interest to tabulate (as a function of q_0, q_1, q_{0r}, q_{1r}, D and T) the critical number of radars needed so that it is immaterial which is attacked first.

6.1.5 Attacks on Defensive Missile Silos

Weiner and McCraith (1970) assume (analogously to Brodheim, Herzer and Russ) that the offense has a stockpile of indefinite size and will continue to attack until I or fewer targets remain undestroyed. However, they introduce an offense option not previously considered — that of attacking the defensive missile silos themselves, as well as the radars and the targets. In order to evaluate this somewhat more complex strategy, it is necessary for them to assume that an undefended target, radar or defensive missile silo is destroyed with probability one, and all defensive missiles fired at offensive weapons intercept them with probability one. All engagements are therefore one-on-one, and any defensive missile can be used to intercept an offensive weapon aimed at any target, radar, or defensive missile silo. All radars must be destroyed to nullify the defense.

The offense attacks in waves. Each wave consists of one weapon directed at each of the D defensive missile silos, or at each of the R radars, or at each of the T targets. The offense cannot assess damage between waves; that is, he does not know which of his weapons were intercepted. How should successive waves be assigned so as to minimize the offensive stockpile required to destroy I or more targets?

The defense strategy is relatively simple and is known to the offense beforehand. If the offense attacks radars, the defense assigns one missile to the defense of a specific radar (unknown to the offense). If the offense attacks targets, the defense assigns I missiles to the defense of I targets (again unknown to the offense). If the offense attacks defensive missile silos, the defense uses half of his unused and undamaged defensive missile stockpile to defend the missile silos of the other half of the stockpile.

Although the restriction is not essential, it is easier to describe the offense strategy when the number of radars, R, and the ratio of total targets to defended targets, T/I, are both integer powers of two. The number of offensive weapons required to ensure that I or fewer targets survive is

$$A = T - IR + D\left(1 + \log_2 R\right) \qquad \text{for} \qquad R \le T/I$$

$$= D\left(1 + \log_2 (T/I)\right) \qquad \text{for} \qquad R \ge T/I$$

What are the offense strategies? If $R \le T/I$, the offense attacks defensive missile silos in $\log_2 R$ waves of D weapons each, reducing the missile stockpile from D to D/R. Then the offense attacks radars in (D/R - I) waves of R weapons each, reducing the missile stockpile to I. Finally, the offense attacks and destroys T - I targets in a single wave of size T. If $R \ge T/I$, the offense attacks defensive missile silos in $\log_2(T/I)$ waves of D weapons each, reducing the missile stockpile from D to D/(T/I). Then the offense attacks targets in D/I(T/I) = D/T waves of T weapons each, reducing the missile stockpile to zero. (Actually, the defense must defend I + 1 targets on each wave, in order to deny the offense its desired destruction until the defensive missile stockpile is exhausted.)

The assumption that the offense can do damage assessment between waves is not present in this model; however, the equally unrealistic assumption that the defense knows the value of I is present. More realistic defense strategies ought to eliminate the latter assumption, perhaps by introducing a tapered defense in which fewer and fewer targets are defended on successive waves. This would, in fact, be a multiple target analogue to some of the defense strategies discussed in Chapter 3.

6.1.6 Attacks on Command and Control Centers

In the preceding section it was shown what modifications of the offense and defense allocation strategies must be made if the defense system itself is vulnerable to attack. This section discusses a closely related problem — that of determining an offense strategy when the targets consist of not only a group of value points but also one or more command and control centers guiding the normal operation of the targets. The offense can either attack the targets or the command and control centers in order to put the targets out of operation. Of course, the two types of attack may not always be equivalent; a destroyed target may be much more difficult to replace than a destroyed command and control center.

Relatively little work has been done on this problem; the following discussion is taken from Piccariello (1962). Assume that one has a set of T undefended targets of identical value; each target has a probability q_0 of surviving a weapon aimed at it. These targets are associated with a group of C command and control centers; each control center has a probability q_{0c} of surviving a weapon aimed at it. If all command and control centers are destroyed, the targets are considered to be completely destroyed. Targets and centers are located sufficiently far apart so that a weapon aimed at any one of them does not affect the others. Assume that the offense has a stockpile of A weapons. How should he allocate weapons to targets and centers in order to minimize the expected fraction of targets surviving? Note that the defense does not enter explicitly here. One may assume either that there is no defense or that the defense has an unlimited stockpile of unreliable missiles and is constrained (say) to one-on-one engagements.

Let x_i, $i = 1,2,\ldots,T$, be the number of offensive weapons allocated to the ith target, and let y_j, $j = 1,2,\ldots,C$, denote the number of weapons allocated to the jth center. The problem, then, is to minimize

$$E(f) = \frac{1}{T}\left(1 - \prod_{j=1}^{C}\left(1 - q_{0c}^{y_j}\right)\right)\sum_{i=1}^{T} q_0^{x_i}$$

subject to the constraint

$$A = \sum_{i=1}^{T} x_i + \sum_{j=1}^{C} y_j = A_T + A_C \quad .$$

If x_i and y_j are not restricted to integral values, Piccariello shows that the minimum strategy is always one in which either $A_T = 0$ or $A_C = 0$. It then follows that the optimum strategies are either $x_i = A/T$ or $y_i = A/C$.

However, this is not necessarily true for strategies restricted to integers; the following simple example demonstrates that the minimum strategy can have $A_C > 0$ and $A_T > 0$. Let $T = 5$, $A = 7$, $C = 2$, $q_{0c} = 1/4$ and q_0 be arbitrary. If all weapons are allocated as evenly as possible to the targets, the expected fraction of targets

surviving is $(2/5)q_0^2 + (3/5)q_0$. If all weapons are allocated to the centers, the expected fraction of targets surviving is
$1 - \left(1 - (1/4)^3\right)\left(1 - (1/4)^4\right) = 319/16384$. Finally, consider the allocation which places one weapon on each target and center. The expected fraction of targets surviving is then
$q_0\left(1 - \left(1 - (1/4)\right)^2\right) = 7q_0/16$. However, for all q_0 it is true that $7q_0/16 < (2/5)q_0^2 + (3/5)q_0$, and for $q_0 < 319/7168$ it is true that $7q_0/16 < 319/16384$. One can show, in fact, that the minimum strategy is the one for which $A_C = 2$, $A_T = 5$.

6.2 MIXTURES OF LOCAL AND AREA DEFENSE MISSILES

The preceding chapters of this monograph have been concerned with a single type of defensive missile. Many realistic defense models postulate the availability of two types of defense missiles of substantially different coverage: a local missile which can defend against weapons directed at a single target, and an area missile which can defend against weapons directed against one of a group of targets in an extended region. This section examines the somewhat complex models of expected target damage and offensive and defensive missile allocation which result from such defense systems.

6.2.1 Defense-Last-Move Models for Area Missiles

One model of the defense of a set of targets of different values using both area and local missiles is presented by Galiano (1967a). Specifically, assume that one has a set of T targets of values $v_1, v_2, \ldots, v_T$. These targets are defended by a stockpile of D_A area missiles which can cover any target in the set, and a stockpile of D_L local missiles which can defend only single targets and must be deployed prior to the attack. The offense has a stockpile of A weapons. The defense knows the value of A, and the offense knows the value of D_A and the allocation of D_L among the targets. The complete allocation of offensive weapons to targets can be seen before the area defensive missiles are allocated (defense-last-move strategy for area missiles). A target is destroyed by an unintercepted weapon aimed at it with probability one, and defensive missiles have perfect reliability.

What are the offense and defense strategies? Galiano assumes that the offense attacks a subset of the targets, each one with a number of weapons proportional to its value. The defense, in turn, is assumed to allocate local missiles in numbers proportional to target value. This local defense is optimum if the object of the offense is to maximize the target damage per weapon expended, and no area missiles are present. It was also suggested as a reasonable one to use (in Section 5.5.2) if the defense does not know the offensive stockpile, although of course he does here. The area missiles defend as much total target value as possible; at each target they defend, the area missiles destroy just enough of the weapons directed at that target so that the local missiles can destroy the remainder of the offense weapons directed at that target.

It is evident that if the parameters are allowed to vary continuously, the only parameter in the offense and defense strategies that must be optimized is the fraction of the total target value which the offense elects to attack. Let A_L denote that part of the offensive stockpile intercepted by local missiles, and $A-A_L$ that part intercepted by area missiles. Then the fraction of value under attack is A_L/D_L, and the fraction of target value destroyed is given by

$$1 - E(f) = \left(1 - \frac{D_A}{A - A_L}\right)\frac{A_L}{D_L} .$$

The optimum value of A_L is

$$A_L^* = A - \left(D_A A\right)^{1/2} ,$$

provided this expression is $\leq D_L$; otherwise $A_L^* = D_L$. In other words, the offense attacks any subset of targets $v_1, v_2, \ldots, v_i$ such that $(v_1 + v_2 + \ldots + v_i)/(v_1 + v_2 + \ldots + v_T)$ is equal to A_L^*/D_L. The expected fraction of targets destroyed is

$$1 - E^*(f) = \left(A - \left(D_A A\right)^{1/2}\right)^2 \Big/ AD_L ,$$

unless $A_L^* = D_L$, in which case $E^*(f) = D_A/(A-D_L)$.

Galiano extends this model somewhat by allowing the defense to locally defend only a fraction, h, of target value. This might occur, for example, if there are large installations of radars associated with local defense which makes it unprofitable locally to defend targets of small value. Assuming that the offense still finds these targets worthwhile to attack, the area defense must be diluted to protect them. The expected fraction of target value destroyed is

$$1 - E(f) = \left(1 - h + h\frac{A_L}{D_L}\right)\left(1 - \frac{D_A}{A - A_L}\right),$$

the optimum value of A_L is

$$A_L^* = A - \left(D_A\left(A + D_L\left(h^{-1} - 1\right)\right)\right)^{1/2} ,$$

and the expected fraction of targets destroyed is

$$1 - E^*(f) = \frac{\left(A + D_L\left(h^{-1}-1\right) - \left(D_A\left(A + D_L\left(h^{-1}-1\right)\right)\right)^{1/2}\right)^2}{\left(A + D_L\left(h^{-1}-1\right)\right) D_L\left(h^{-1}\right)} .$$

The optimum value of h can, in principle, be calculated from a knowledge of the relative costs of defense missiles and radar installations.

Galiano's model is slightly generalized by Kooharian, Saber, and Young (1969). Specifically, they assume that it requires either t local missiles or s area missiles to destroy a single incoming weapon. It is apparent that if the parameters are allowed to vary continuously, this is equivalent to the above with D_A replaced by D_A/s and D_L replaced by D_L/t. Thus

$$1 - E(f) = \left(1 - \frac{D_A}{s\left(A-A_L\right)}\right) \frac{tA_L}{D_L} ,$$

and the optimum value of A_L is

$$A_L^* = A - \left(D_A A/s\right)^{1/2} .$$

The offense attacks a subset of the targets $v_1, v_2, \ldots, v_i$ such that $(v_1+v_2+\ldots+v_i)/(v_1+v_2+\ldots+v_T)$ is equal to tA_L^*/D_L. If only a fraction h of the targets are defended by local missiles,

$$1 - E(f) = \left(1 - h + h\frac{tA_L}{D_L}\right)\left(1 - \frac{D_A}{s\left(A-A_L\right)}\right) .$$

The authors reparameterize the problem using the variables $\tau = D_L/V$ (number of local missiles defending a target of unit value) and $\rho = (tA_L/D_L)V/s(A-A_L)$ (target value saved per area missile employed); the offense optimization is then carried out on ρ instead of A_L. The authors consider mixtures of weapons indexed by the ratio s/t, assuming that the defense can discriminate between weapon types prior to commitment of missiles. They show that the offense will use weapons with high s/t ratios against locally undefended targets, and weapons with low s/t ratios against locally defended targets (with at most one weapon type used against both target types).

6.2.2 Preallocation Models for Area Missiles

Apparently, no one has determined the optimum defense of a set of targets of different values using both local and area missiles when the defense does not know the offense allocation to targets but can identify which target is being attacked in time to intercept it with an area defense missile, and the defense uses a preallocation strategy for area missiles. However, Galiano (1968) analyzes this problem for targets of equal value, assuming the defense has a stockpile of d_a area missiles per target and d_ℓ local missiles per target, and the offense has a stockpile of a weapons per target. Each side knows the stockpile size of the other side; furthermore, a target is destroyed by an unintercepted weapon aimed at it with probability one, and defensive missiles have perfect reliability.

The most important differences between Galiano's model and the earlier ones discussed in this section is that offense and defense allocations are allowed to vary continuously, and that a target is destroyed if and only if the weapon allocation exceeds the missile allocation at that target (even by an infinitesimal amount). In short, Galiano introduces a continuous Blotto game as described in Section 4.3.6. Because of the bias in favor of the offense, this continuous Blotto game cannot be readily compared with the allocations elsewhere in this section.

It is evident that the local missiles should be distributed evenly among the targets. Therefore, in order to destroy a target, the weapon allocation must exceed the area missile allocation by more than d_ℓ. Two cases must be distinguished — offense dominance $\left(\text{when } d_a\left(a - \left(d_\ell/2\right)\right) \le \left(a-d_\ell\right)^2\right)$ and defense dominance $\left(\text{when } d_a\left(a - \left(d_\ell/2\right)\right) \ge \left(a-d_\ell\right)^2\right)$. If the offense is dominant, then he attacks a typical target with a_i weapons, where a_i is a real number selected from the uniform probability density function between d_ℓ and $2a - d_\ell$. The defense defends a typical target with area missiles with probability $d_a/(a-d_\ell)$; if defended, a target is allocated d_i area missiles, where d_i is a real number selected from the uniform probability density function between 0 and $2a - 2d_\ell$. The expected fraction of targets saved is $E(f) = d_a/2(a-d_\ell)$. If the defense is dominant, then he defends a typical target with area missiles with probability $(1/d_\ell)\left(d_a^2 + 2d_a d_\ell\right)^{1/2} - (d_a/d_\ell)$; if defended, a target is allocated d_i area missiles, where d_i is a real number selected from the uniform probability density function between 0 and $d_a + \left(d_a^2 + d_a d_\ell\right)^{1/2}$. The offense attacks a typical target with probability $2a \Big/ \left(d_a + 2d_\ell + \left(d_a^2 + 2d_a d_\ell\right)^{1/2}\right)$; if attacked, a target is allocated a_i weapons, where a_i is a real number selected from the uniform probability density function between d_ℓ and $d_\ell + d_a + \left(d_a^2 + d_a d_\ell\right)^{1/2}$. The expected fraction of targets saved is $E(f) = 1 - \left(a/d_\ell^2\right)\left(d_a + d_\ell - \left(d_a^2 + 2d_a d_\ell\right)^{1/2}\right)$. When d_ℓ is set equal to zero, these results reduce to the continuous Blotto game discussed in Section 4.3.6.

It should also be pointed out that the equal-value case can be regarded as another generalization of the Matheson game of Chapter 4 and can be solved by linear programming (see Section 4.3.4).

6.2.3 Models Involving Area Missiles of Limited Range

In the preceding two sections, area defense missiles were assumed to cover a single region encompassing all the targets. This section generalizes this model to allow for two or more different regions, either partially overlapping each other or completely independent (sometimes a reasonable approximation to regions which overlap to a limited extent). As one might expect, it is impossible to give either the expected value destroyed or the allocation of area vs. local missiles in simple formulas; one must resort to linear programming algorithms or to simulation.

Consider a nested allocation problem involving both local and area missiles. Instead of a set of targets in a single region covered by area missiles, assume that there are several nonoverlapping regions to be defended, each containing several point targets of different values. Local missiles can defend only a single target; area missiles can defend any of the targets within a region. If the defense has a stockpile of D_A area missiles and D_L local missiles, and the offense has a stockpile of A weapons, how should area missiles be allocated to regions and local missiles to targets to minimize the expected value of targets lost, assuming the most damaging offensive attack against that allocation?

As usual, one must make further assumptions about the capability of the area missiles. Consider two models: an area defense strategy for area missiles (random arrival), which may be called a separable model, and a defense-last-move strategy for area missiles, which may be called a nonseparable model. Unintercepted weapons destroy targets at which they are aimed with probability unity; the reliability of area and local missiles is given by ρ_A and ρ_L. One-on-one missile engagements are assumed.

Unfortunately, there appears to be no known algorithm for calculating the minimax strategies and expected value of targets destroyed in this nested allocation model; one must enumerate all cases and search for the minimax directly. One can show by a numerical example that one cannot find the optimum allocation by breaking the problem into two parts, within-sector and between-sector, and optimizing local and area missile allocations separately. As might be expected, the direct search for the minimax is even more laborious in the nonseparable model than in the separable one; the defense allocation of area missiles to targets after observing the offensive allocation must also be taken into account.

If one is willing to change the damage criterion, an approximate solution to the nested allocation problem can be found. Suppose that the offense has a stockpile of indefinite size, and wishes to allocate weapons to targets so as to minimize his cost in terms of weapons expended per unit of value destroyed. Given a reasonably balanced local missile allocation to targets, it is a relatively straightforward task for the defense to allocate area missiles among sectors so that the offense minimum cost per unit value destroyed is the same for every sector. The local missiles, moreover, can be allocated to targets within a sector so that the offensive minimum cost per unit value destroyed is the same for every target within a sector, ignoring the contribution of the area missiles to the defense. A few numerical examples have been worked out showing that the allocations using this criterion are similar to the minimax allocations. It would be of great value to the missile defense designer to know the mathematical conditions under which this substitute criterion generates allocations which are good approximations to the minimax allocations.

Miercort and Soland (1971) consider an even more general pattern of area missile coverage — several partially-overlapping regions superimposed upon a set of point targets of different values. The r most valuable targets are assigned local missiles for point defense. Both the area and local missile allocations and coverages are known to the offense prior to the attack. Miercort and Soland assume reliable defensive missiles ($\rho_L = \rho_A = 1$); on the other hand, they allow a general function of target damage by unintercepted weapons. If x_j unintercepted weapons attack target j, then the expected value destroyed is given by $v_j f(x_j)$, where $f(x_j)$ is a concave and nondecreasing function with $f(0) = 0$. An area defense strategy for area missiles (controlled arrivals) is assumed; that is, the defense uses his area missiles to counter any incoming weapons, and then uses his local missiles after the area missile stockpile for that region has been exhausted. All missile engagements are one-on-one.

No attempt is made to optimize the defensive allocation between local and area missiles; Miercort and Soland restrict themselves to finding that allocation of an offensive weapon stockpile of size A to point targets which will maximize the expected value destroyed. Their optimization is presented as a branch-and-bound algorithm; the reader is referred to Miercort and Soland for details.

6.3 MODELS FOR LOCAL AND AREA MISSILES INVOLVING COSTS

The models in this section differ from those in the previous section in that the defense is given a fixed budget to divide as he pleases among local and area missiles. Thus, the optimization involves this division, as well as the conduct of the defense if attacked. Sections 6.3.1 - 6.3.5 are the only ones in this monograph in which cost is considered; it is appropriate to consider cost in this case, since the defense potentially has resources of two types. In each case in this section, cost is held constant, while the defense tries to minimize the value destroyed.

Let a denote the number of offensive weapons allocated to each target in the set, and let d be the number of area missiles per target which the defense can purchase if he selects a pure area-missile defense. It is convenient (although not essential) to restrict a and d to integral values. Let $K > 1$ be the ratio of the cost of an area missile to that of a local missile. The defense has the option of providing d-j area missiles per target and jK local missiles per target, for $j = 0,1,\ldots,d$. The defense is assumed to know the value of a, and the offense is assumed to know the values of d and j selected by the defense, as well as K. The offense has the option of attacking all targets with a weapons apiece or a fraction f of the targets with a/f weapons apiece, $0 < f < 1$. These offensive strategies can be indexed using the letter i in various ways; for example, $f = i/jK$ for $0 < i \le jK$, or $f = a/(a+i)$, for $i = 0,1,2,\ldots$. By attacking a subset of the targets, the offense avoids some of the local defense weapons.

It appears quite difficult to carry out an analysis of this missile allocation problem except in the simplest situations. Specifically, assume that a one-on-one defense is used, and that the targets are sufficiently far apart so that a weapon aimed at one target affects no other targets. Assume that q_0, the probability that a target survives an unintercepted weapon directed at it, is zero, and that q_1, the probability that a target survives an intercepted weapon directed at it, is unity.

It should be noted at the outset that the defense can save all the targets if $d \geq a$ by using all area missiles and intercepting each attacker as it arrives. (This can be called an attacker-oriented defense.) Therefore the analyses below will assume $d < a$.

The allocation between local and area missiles depends strongly upon the assumptions made about the effectiveness of the area missiles. Three different models are considered:

1. The complete allocation of offensive weapons to targets can be seen before area missiles are allocated to them (defense-last-move strategy for area missiles).
2. As each weapon arrives, the defense assigns an area missile to it, not knowing which target is being attacked, until the stockpile of area missiles is exhausted. The offense cannot control the arrival-order of his weapons (area defense strategy for area missiles, random arrivals).
3. As each weapon arrives, the defense assigns an area missile to it, not knowing which target is being attacked, until the stockpile of area missiles is exhausted. The offense is able to control the arrival-order of his weapons (area defense strategy for area missiles, controlled arrivals).

6.3.1 A Defense-Last-Move Model

The defense-last-move strategy pertaining to case 1 for area missiles is considered first. It is unlikely that this situation will often arise in practice; nevertheless, it furnishes an upper bound for defense capability. Moreover, should area defense missiles not be useful in this model, it seems unlikely that they can be profitably employed when less is known about the attack.

The defense has available d-j area missiles per target (averaged over the original set of targets) and jK local missiles per target. Note that these area missiles are not assigned to the defense of any particular target; however, it is convenient to keep track of offensive weapons and defensive missiles on a per-target basis. If the offense attacks a fraction i/jK of the targets with ajK/i weapons apiece, then jK of these weapons will be destroyed by local defense missiles at each target attacked, leaving (a-i)jK/i weapons at each target attacked to be assigned to area defense missiles. In other words, there are a-i weapons per target (averaged over the original set of targets, not just the attacked targets) to be assigned to area defense missiles. If the defense has the last move, it can save a fraction (d-j)/(a-i) of the targets being attacked using its area missiles. The expected fraction of targets destroyed if the defense mixes area and local missiles using strategy j and the offense attacks a fraction i/jK of the targets is then given by

$$1 - E(f) = (i/jK)\Big(1 - (d-j)/(a-i)\Big) .$$

This equation is equivalent to the one developed by Galiano (1967a) in Section 6.2.1, if one notes that $D_A = T(d-j)$, $D_L = TKj$, $A = Ta$ and $A_L = (i/jK)d_L = iT$.

The offense can choose i after observing the defensive choice of j; therefore, the goal of the defense is to choose j so that the maximum damage the offense can do is minimized. In other words, one must find those strategies i and j yielding

$$1 - E^*(f) = \min_j \max_i \left(1 - E(f)\right) .$$

Permitting fractional allocations, the optimum strategies and the corresponding $E^*(f)$ can be easily found by straightforward techniques of differential calculus. If $1 \leq (a/d) \leq K$, the optimum defense strategy is given by $j = d(1 - (d/a))$; in other words, one allocates d^2/a area missiles per target. The offense attacks a fraction $i/jK = a/dK$ of targets, and the expected fraction of targets destroyed is

$$1 - E^*(f) = (a/dK) \left(1 - (d/a)\right) = (a-d)/dK .$$

Note that this is the same as the corresponding formula by Galiano (1967a) in Section 6.2.1, if the above substitutions are made. However, this formula is substantially simpler because $j = d(1 - (d/a))$.

As has been mentioned above, if $a/d \leq 1$, one can deploy only area defense missiles and the entire set of targets can be saved. Furthermore, if $a/d \geq K$, all area defense missiles should be used; the expected fraction of targets destroyed is then $1 - (d/a)$. It is interesting that it is necessary to mix area and local missiles only in some middle range for a/d. The reason for this fact is that when a is large the defense cannot afford to give up the greater efficiency of area missiles, and if a is small the defense doesn't need local missiles anyway. Note, however, that in the latter case, one could also use a mixture of area and local missiles and still save all the targets defended.

6.3.2 A Model in Which the Offense Arrival Order is Random

The area defense strategy for area missiles (random arrivals), pertaining to case 2, is considered next. The defense has d-j area missiles per target (averaged over the original set of targets) and jK local missiles. The defense attacks offensive weapons in the order of their arrival, first using area missiles; when this part of his stockpile is exhausted, he uses local missiles. The area missiles are assigned in ignorance of the targets the weapons are aimed at; each local missile defends its assigned target against an offensive weapon directed against it. The offense attacks a fraction $a/(a+i)$ of the targets in the set with $(a+i)$ weapons apiece, $i = 0,1,2,\ldots$. Weapon arrivals are random with respect to the targets the weapons are directed against.

If the defense mixes area and local missiles according to strategy j and the offense attacks a fraction $a/(a+i)$ of the targets, the expected fraction of targets destroyed can be derived with the aid of the hypergeometric probability density function:

$$1 - E(f) = \frac{a}{a+i} \sum_{m=0}^{C} \binom{a+i}{m} \binom{aT - (a+i)}{T(d-j) - m} \Big/ \binom{aT}{T(d-j)} ,$$

where T is the number of targets and C is the minimum of T(d-j) and (a+i) - (jK+1). Besides being tedious to compute, this expression contains the additional variable T. If one is willing to assume that the number of offensive weapons to be engaged by local missiles at the ith target is independent of the number of offensive weapons to be engaged by local missiles at the jth target, then a binomial argument can be used to derive E(f).

The probability that an offensive weapon is intercepted by an area missile is (d-j)/a; if there are (a+i) weapons aimed at an attacked target, the probability that exactly m are intercepted by area missiles is approximated by

$$P_m = \binom{a+i}{m} \left(\frac{d-j}{a}\right)^m \left(1 - \frac{d-j}{a}\right)^{a+i-m} .$$

(Recall that $d < a$ has been assumed, so that the above is meaningful for any $0 \le j \le d$.) In order for the target to be saved, there must be jK or fewer of the (a+i) weapons that were not intercepted; that is, $a + i - m \le jK$, or $m \ge a + i - jK$. Thus, the probability that a random target is destroyed is equal to the summation of binomial terms:

$$P = \frac{a}{a+i} \sum_{m=0}^{a+i-jK-1} P_m .$$

Again using the independence argument above, P is equated to 1 - E(f). This approximation is likely to be a very good one as long as T is large.

The offense can choose i after observing the defensive choice of j; therefore, the goal of the defense is to choose j so that the maximum damage the offense can do is minimized. In other words, one must find those strategies i and j yielding

$$1 - E^*(f) = \min_j \min_i \left(1 - E(f)\right) .$$

Unfortunately, there is no simple analytic solution to this problem; one must directly search the matrix of E(f), viewed as a function of i and j.

Table 1, on the next page, may give the reader some insight into the behavior of the optimum i and j and the corresponding E*(f) for moderate values of a, d and K.

TABLE 1

OPTIMUM STRATEGIES AND THE EXPECTED FRACTION OF TARGETS DESTROYED

d = 2, a = 3

K	E*(f)	i	j
1	.704	0	0
2	.474	2	1
3	.354	4	1
5	.272	8	2

d = 2, a = 4

K	E*(f)	i	j
1	.937	0	0
2	.738	0	1
3	.554	2	1
5	.364	7	2

d = 2, a = 5

K	E*(f)	i	j
1	.990	0	0
2	.942	0	1
3	.714	2	2
5	.455	6	2

d = 4, a = 5

K	E*(f)	i	j
1	.663	0	1
2	.417	5	2
3	.304	9	2
5	.200	18	2

d = 4, a = 6

K	E*(f)	i	j
1	.891	0	1
2	.558	4	3
3	.399	8	3
5	.256	16	3

d = 4, a = 8

K	E*(f)	i	j
1	-	-	-
2	.807	1	3
3	.573	5	3
5	.366	13	3

d = 4, a = 12

K	E*(f)	i	j
1	-	-	-
2	-	-	-
3	.923	1	4
5	.571	9	4

d = 8, a = 9

K	E*(f)	i	j
1	.628	0	1
1½	.475	5	2
2	.385	10	2
3	.279	19	3
5	.183	36	5

d = 8, a = 10

K	E*(f)	i	j
1	.828	0	3
1½	.607	4	4
2	.481	8	4
3	.341	17	5
5	.217	34	6

d = 8, a = 12

K	E*(f)	i	j
1	-	-	-
1½	-	-	-
2	.623	6	6
3	.435	15	7
5	.272	31	7

d = 8, a = 16

K	E*(f)	i	j
1	-	-	-
1½	-	-	-
2	.868	2	7
3	.603	10	7
5	.375	26	7

d = 16, a = 17

K	E*(f)	i	j
1	.605	1	1
1¼	.540	5	2
1½	.458	10	2
2	.371	20	3
3	.268	38	4
5	.173	73	5

d = 16, a = 18

K	E*(f)	i	j
1	.769	0	4
1¼	.650	4	4
1½	.564	9	4
2	.440	18	6
3	.312	35	7
5	.197	68	8

d = 16, a = 20

K	E*(f)	i	j
1	-	-	-
1¼	.799	2	8
1½	.682	7	10
2	.530	15	11
3	.368	32	12
5	.229	64	14

d = 16, a = 24

K	E*(f)	i	j
1	-	-	-
1¼	-	-	-
1½	.870	3	14
2	.670	11	14
3	.459	27	14
5	.283	59	14

From this table, one can conclude that the offense strategy i is approximated by $dK - a + 1$; however, as $d \to \infty$, $K \to \infty$ and $a \to d$, the errors in this formula become substantial (for example, when $d = 16$, $a = 17$ and $K = 5$, i is predicted to be 64 but is actually 73). If $a > 3d/2$, the defense strategy j is approximated by $d-1$, regardless of the value of K. These rules of thumb are helpful in narrowing the area of search in the E(f) matrix.

6.3.3 A Model in Which the Offense Can Control His Arrival Order

The area defense strategy for area missiles (controlled arrivals), pertaining to case 3, is the final model to be discussed in this section. The defense has d-j area missiles per target (averaged over the original set of targets) and jK local missiles. The defense attacks offensive weapons in the order of their arrival, first using area missiles; when this part of his stockpile is exhausted, he uses local missiles. The area missiles are assigned in ignorance of the targets the weapons are aimed at; each local missile defends its assigned target against an offensive weapon directed against it. The offense attacks a fraction $a/(a+i)$ of the target, $i = 0,1,2,\ldots$. The key difference between this model and the one analyzed previously is that the offense can control the order of arrival of his weapons on targets.

The offense exhausts the area missile stockpile with the first $(d-j)T$ weapon arrivals, and then attacks as many targets as possible with $(jK+1)$ weapons apiece. If a is sufficiently large the offense can choose $i = 0$ and destroy all the targets. If not, he selects i to satisfy the following equation:

$$d - j + \frac{a}{a+i}(jK+1) = a \ .$$

From these two cases one has

$$i = \max\left(0, \frac{a(jK+1)}{a-d+j} - a\right) \ .$$

The expected fraction of targets destroyed by the optimal attack is

$$E(f) = a/(a+i) = \min\left(1, \frac{a-d+j}{jK+1}\right) \ .$$

It is easily seen that $(a-d+j)/(jK+1)$ is a monotone function of j for $j \geq 0$, so that the minimum of the function is assumed at either $j = 0$ or $j = d$. Thus,

$$\min_j \max_i E(f) = \min\left(1, a-d, a/(dK+1)\right) \ .$$

(Recall that $a > d$.) It is interesting that if $a - d = a/(dK+1)$, so that $a - d = 1/K$, then any choice of j is optimal. Otherwise (unless the above minimax is 1 and no defense is possible) the optimal defense is one of the two extremes: all local or all area missiles, according to the inequalities $(1/K) > (a-d)$ and $(1/K) < (a-d)$.

6.3.4 A Comparison of Models

It is interesting to calculate the expected fraction of targets destroyed as a function of the effectiveness of the area missiles, under the assumption that the optimum mix of local and area missiles is used. Let $a = 4$, $d = 3$ and $K = 2$.

1. Defense-last-move strategy for area missiles: the defense allocates $d^2/a = 9/4$ area missiles and $6/4$ local missiles per target, the offense attacks a fraction $a/dK = 2/3$ of the targets, and the expected fraction of targets lost is 1/6. The nearest integer solution is to allocate two area and two local missiles per target: the offense attacks a fraction 0.586 of the targets and the expected fraction lost is 0.171.
2. Area defense strategy for area missiles (random arrivals): the defense allocates two area and two local missiles per target, the offense attacks 4/7 of the targets with 7 weapons apiece, and the expected fraction of targets lost is 0.442.
3. Area defense strategy for area missiles (controlled arrivals): the defense allocates no area and six local missiles per target, the offense attacks 4/7 of the targets with 7 weapons apiece, and the expected fraction of targets lost is $4/7 = 0.572$.

Note that under typical conditions, the defense can achieve the same results as those of case 2, even if the offense can control the order of his arrivals, by allocating his d-j area missiles at random among the a arriving weapons. This is an attacker-oriented defense such as discussed in Section 4.6.2. and of course can be applied only if no physical limitation precludes it.

6.3.5 An Allocation Model for Targets of Unequal Value

This subsection generalizes two of the models discussed earlier to targets of different values. Assume that one has a set of T point targets with values $v_1 \geq v_2 \geq \ldots \geq v_T$, where $\Sigma v_i = V$. These targets can be defended by area missiles with a kill-probability of ρ_A, or local missiles with a kill-probability of ρ_L. The defense uses a shoot-look-shoot strategy; if a missile fails, it is replaced by another one. Each area missile is the economic equivalent of K local missiles. Area missiles can defend any target but cannot distinguish which targets the offensive weapons are aimed at. As each offensive weapon arrives, area missiles are launched at it until it is destroyed or the stockpile of area missiles is exhausted. Only when no area missiles remain are local missiles employed. If a weapon is not intercepted, it destroys the target at which it has been aimed.

The defense has a budget which allows him to purchase Td area missiles, T(d-1) area missiles and KT local missiles, ..., or KdT local missiles. The offense has a stockpile of unspecified size, and wishes to allocate weapons so as to minimize his cost in terms of weapons expended per unit of value destroyed. The offense is assumed to have complete control over the arrival sequence of the attack.

It is not difficult to show that all targets above a certain minimum value should be defended. If the defense elects to place local missiles at a target, he should place a number of missiles proportional to the value of the target: $d_m = fv_m$, $m = 1,2,\ldots,i$. (This gives fractional allocations in general; however, one can also give a search technique leading to integral allocations.) The defense strategy then consists of two decisions only: the value of f, and the choice of i. The sum of d_m over the defended targets determines the number of local missiles used; the remainder of the defense stockpile is then invested in area missiles.

The offense employs a strategy based upon defensive stockpile exhaustion. Assume that the defense has allocated T(d-j) missiles to area defense, and is defending targets $v_1, v_2, \ldots, v_i$ with $d_1, d_2, \ldots, d_i$ local missiles, where $d_1 + d_2 + \ldots + d_i = jKT$. The offense first allocates $\rho_A T(d-j)$ weapons (on the average) to exhaust the area defense missiles using a shoot-look-shoot doctrine; then he allocates $\rho_L d_m$ weapons to the target of value v_m (if he has decided to attack this target at all) in order to exhaust the local defense missiles; finally he allocates one weapon apiece to each target with no local defense and each target with an exhausted local defense.

What are the optimum defense and offense strategies? It is possible to prove that the optimum defense strategy takes one of two forms:

1. If $\rho_A \geq K\rho_L$, use area missiles exclusively.
2. If $\rho_A < K\rho_L$, use a mixture of area and local missiles. The number T_d of targets to be defended by local missiles is that value of i which maximizes the following expression:

$$C(i) = \frac{T\left(\rho_A d + 1\right) - i}{V - \left(1 - \left(\rho_A / K\rho_L\right)\right)\left(v_1 + v_2 + \ldots + v_i\right)} .$$

Let V_d denote the total value of the T_d locally defended targets. The value of f is found from the equation

$$f = C(T_d)/\rho_L .$$

Once f is determined, the total number of local missiles is given by fV_d and the number of area missiles is given by $Td - fV_d/K$.

The offense strategies are simple. If $\rho_A \geq K\rho_L$, attack all targets in the set. If $\rho_A < K\rho_L$, either attack all targets in the set, or attack all targets not defended by local missiles; the payoff is the same in either case. The number of weapons expended per unit of value destroyed is given by $C(T_d)$.

6.4 OFFENSE AND DEFENSE STRATEGIES FOR A GROUP OF AREA TARGETS

The mathematical models introduced in this chapter and the two preceding ones have been designed specifically for point targets, that is, for targets of small size relative to the lethal radius of an offensive weapon. It is easy to specify a damage model for point targets: if a single weapon destroys the target with probability p, a group of n weapons independently directed at the target destroys it with probability $1 - (1-p)^n$. Unfortunately, the corresponding damage law for an area target is much more difficult to specify in a simple form. In Chapter 2, the expected damage to a Gaussian target (or to a uniform-valued circular target) by n weapons having a cookie-cutter damage function is presented in the form of an integral which cannot be analytically evaluated. Consequently, it appears quite difficult to derive offensive and defensive allocation models for multiple area targets analogous to the models for multiple point targets given in Chapters 4 and 5. Very little research has been carried out on this problem.

Can a simple analytic approximate damage law be postulated for area targets? In Chapter 2, the square root damage law for undefended Gaussian targets can be derived if one assumes that

1. the number of offensive weapons, n, approaches infinity and the lethal radius, R, of each weapon approaches zero in such a way that $nR^2 = c$, and
2. the probability density function of weapon impact points is not Gaussian but optimum.

Although the square root damage law becomes less plausible for defended Gaussian targets, it may be a useful approximation. In particular, it seems reasonable when the defense is unable to determine the aimpoints of individual weapons before committing missiles to them. Assume that the ith target in a defended group has value v_i, distributed according to a circular Gaussian value density function with variance σ_T^2. Using the square root damage law, it is easy to construct a function specifying the expected value saved at the ith target if it is attacked by a_i offensive weapons and has d_i missiles (each with perfect reliability) assigned to its defense. If $a_i > d_i$,

$$E\left(i,a_i,d_i\right) = v_i\left(1 + \left(a_i-d_i\right)^{1/2} R/\sigma_T\right) \exp\left(-\left(a_i-d_i\right)^{1/2} R/\sigma_T\right) \;;$$

if $a_i \leq d_i$,

$$E\left(i,a_i,d_i\right) = v_i \;.$$

Assuming that the offense has the last move, the optimum offense and defense strategies for a group of targets of different values can be derived by the heuristic methods of Pugh (1964) discussed in Chapter 5. The details of this procedure can be found in Goodrich (1968). It would be of interest to generalize Goodrich's allocation model to defensive missiles with reliability less than unity.

Note that this model assumes that, when $a_i > d_i$, weapons are intercepted at random in the original attack of size a_i. If the defense can determine where each weapon will impact on the target before committing a defensive missile to it, it should be possible to design an improved defense (perhaps one that defends a subregion of the target and abandons the rest). Defense models of area targets taking advantage of impact point information remain to be investigated.

Under certain circumstances, generalized area target models can be handled approximately by the methods of Chapters 4 and 5.

6.5 SUMMARY

This chapter attempts to model the offense-defense problem in more realistic detail than was considered in the previous two chapters. Specifically, two important defense problems are modeled: the vulnerability of the defense system itself to attack, and the allocation of high-cost area missiles and low-cost local missiles to the defense of a group of identical targets. Models of both types are widely scattered through both the unclassified and classified literature, principally in government and defense contractor reports, and the object of this chapter is more to call the readers' attention to the range of problems involved than to survey systematically the state of the art.

In general, analytic solutions appear impossible to obtain except in situations so simplified that nearly all realism is gone; simulation is usually required. However, simulation is also unsatisfactory because the extremely large number of variables needed to characterize the model makes sensitivity analyses and exploration of alternatives tedious. At the end of the chapter, the question of finding offense and defense strategies for area targets (not specifically considered since Chapter 2) is raised.

BIBLIOGRAPHY

The following 138 references are all unclassified. The entry in parentheses at the end of each reference indicates the place(s) in the monograph where it is discussed, for example, (5.3.2) indicates that the reference is discussed in Section 5.3.2.

Abramowitz, M. and Stegun, J. A. (editors) (1964), Handbook of Mathematical Functions, National Bureau of Standards. (2. Introduction, 2.1.2, 2.4.3, 2.5.1)

Abramson, L. R. and Shapiro, M. (no date), Defense With Damage Assessment, Technical Report T-6/322, Electronics Research Laboratories, Columbia University. (4.5.1)

Analytic Services Corporation (1968), Final Report on Preferential Strategies for Urban Ballistic Missile Defense, Falls Church, Virginia. (4.4.3)

Bell Aircraft Corporation (1956), Tables of Circular Normal Probabilities, Report 02-949-106, Operations Analysis Group, Dynamics Section, Bell Aircraft Corporation, Buffalo, New York. (2.1.2)

Bellman, R. and Dreyfus, S. (1962), Applied Dynamic Programming, Princeton University Press. (5.2.1)

Beyer, W. H. (editor) (1966), Handbook of Tables for Probability and Statistics, Chemical Rubber Co. (2.1.3)

Blackett, D. W. (1954), Some Blotto Games, Naval Research Logistics Quarterly 1, p. 55-60. (4.3.6)

Blischke, W. R. and Halpin, A. (1966), Asymptotic Properties of Some Estimators of Quantiles of Circular Error, J. Amer. Statist. Assoc. 61, 618-632. (2.6.2)

Bradley, E., (1965), The Convex Hull of a Random Set of Points, Biometrika 52, 331-343. (2.6.2)

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Breaux, H. J. (1968), A Note on Two Coverage Problems for Multiple Shots, Opns. Res. 16, 1239-1242. (2.4.3, 2.4.4)

Breaux, H. J. and Mohler, L. S. (1971), A Computational Procedure for a Class of Coverage Problems for Multiple Shots, Opns. Res. 19, 636-644. (2.4.3, 2.4.4)

Brennan, L. E. and Reed, I. S. (1965), A Recursive Method of Computing the Q-Function, IEEE Transactions on Information Theory 11, 312-313. (2.1.2)

Bressel, C. N. (1971), Expected Target Damage for Pattern Firing, Opns. Res. 19, 655-667. (2.5.1)

Brodheim, E., Herzer, I. and Russ, L. M. (1967), A General Dynamic Model for Air Defense, Opns. Res. 15, 779-796. (4.7.4, 6.1.4)

Bronowski, J. and Neyman, J. (1945), The Variance of the Measure of a Two-Dimensional Random Set, Ann. Math. Statist. 16, 330-341. (2.4.1)

Burington, R. S. and May, D. C. (1953), Handbook of Probability and Statistics With Tables, Sandusky, Ohio, Handbook publishers. (2.1.2)

Burr, S. A. and Graham, R. L. (1970), On a Class of Equivalent Linear and Nonlinear Integer Programming Problems, Colloquia Mathematica Societatis Janos Bolyai 4 (Combinatorial Theory and Its Applications), Hungary, 199-211. (4.6.2, 5.1, 5.8)

Cacoullos, T. and DeCicco, H. (1967), On the Distribution of the Bivariate Range, Technometrics 9, 476-480. (2.6.2)

Charnes, A. (1953), Constrained Games and Linear Programming, Proc. Nat. Acad. Sci. U.S.A. 38, 639-641. (4.3, 4.3.1, 4.3.4)

Charnes, A. and Cooper, W. W. (1965), A Note on the 'Fail-Safe' Properties of the "Generalized Lagrange Multiplier Method", Opns. Res. 13, 674-677. (5.2.2)

Cooper, J. N. and Restrepo, R. A. (1967), Some Problems of Attack and Defense, Soc. Indus. Appl. Math. Rev. 9, 680-691. (5.6.2)

Daniels, H. E. (1952), The Covering Circle of a Sample From a Circular Normal Distribution, Biometrika 39, 137-143. (2.6.2)

Danskin, J. M. (1966), The Theory of Max-Min, With Applications, Soc. Indus. Appl. Math. J. 14, 641-664. (5.4.1)

Danskin, J. M. (1967), The Theory of Max-Min, Springer-Verlag, New York. (5.1, 5.4.1)

den Broeder, G. G., Ellison, R. E. and Emerling, L. (1959), On Optimum Target Assignments, Opns. Res. 7, 322-326. (5.1)

DiDonato, A. R. and Jarnagin, M. P. (1960), Integration of the General Bivariate Gaussian Distribution over an Offset Ellipse, N.W.L. Report 1710, U.S. Naval Weapons Laboratory, Dahlgren, Virginia. (2.1.4)

DiDonato, A. R. and Jarnagin, M. P. (1962), A Method for Computing the Generalized Circular Error Function and Circular Coverage Function, N.W.L. Report 1768, U.S. Naval Weapons Laboratory, Dahlgren, Virginia. (2.1.2, 2.1.3)

Dresher, M. (1961), Games of Strategy: Theory and Applications, Prentice-Hall. (5.6.1, 5.6.2)

Duncan, R. L. (1964), Hit Probabilities for Multiple Weapons Systems, Soc. Indus. Appl. Math. Rev. 6, 111-114. (2.3.3, 2.5.2, 2.5.4)

Dym, C. L. and Schwartz, R. E. (1969), Optimum Pre-Attack Targeting, Research Paper P-546, Institute for Defense Analyses, Arlington, Virginia. (3.2.4, 5.1, 5.8)

Eckler, A. R. (1969), A Survey of Coverage Problems Associated with Point and Area Targets, Technometrics 11, 561-589. (2. Introduction)

Eisen, D. (1967), Defense Models I: Basic Analytical Approaches to Ballistic Missile Defense Problems, Paper 3, Lambda Corporation, Arlington, Virginia. (5.5.1)

Esperti, R. V. (1960), Tables of the Elliptical Normal Probability Function, Defense Systems Division, General Motors Corporation, Detroit, Michigan. (2.1.3)

Everett, H. (1963), Generalized Lagrange Multiplier Method for Solving Problems of Optimum Allocation of Resources, Opns. Res. 11, 399-417. (5.2.2)

Everett, H. (1965), Comments on Preceding Note (by Charnes and Cooper), Opns. Res. 13, 677-678. (5.2.2)

Everett, H. (1968), Defense Models XI: Exhaustion of Interceptor Supplies, Paper 17, Lambda Corporation, Arlington, Virginia. (3.2.3)

Fejes Tóth, L. (1953), Lagerungen in der Ebene auf der Kugel und im Raum, Springer-Verlag, Berlin. (2.2.3, 2.5.1)

Galiano, R. J. (1967a), Defense Models II: Simple Analytic Models for Terminal and Preferential Area Defense, Paper 4, Lambda Corporation, Arlington, Virginia. (6.2.1, 6.3.1)

Galiano, R. J. (1967b), Defense Models III: Preferential Defense, Paper 5, Lambda Corporation, Arlington, Virginia. (4.3.6)

Galiano, R. J. (1968), Defense Models VI: Strategies of Preferential Defense in Combination With Fixed Terminal Defense, Paper 8, Lambda Corporation, Arlington, Virginia. (6.2.2)

Galiano, R. J. (1969), Defense Models XVI: A Simple Analytic Model for Preallocated Defense, Paper 30, Lambda Corporation, Arlington, Virginia. (4.3.6)

Galiano, R. J. and Everett, H. (1967), Defense Models IV: Some Mathematical Relations for Probability of Kill; Family of Damage Functions for Multiple Weapon Attacks, Paper 6, Lambda Corporation, Arlington, Virginia. (2.4.1, 2.5.4)

Germond, H. H. (1950), The Circular Coverage Function, Rand Report RM-330, The Rand Corporation, Santa Monica, California. (2.2.1)

Gilbert, J. P. and Mosteller, F. (1966), Recognizing the Maximum of a Sequence, J. Amer. Statist. Assoc. 61, 35-73. (3.3.1, 3.3.2)

Gilliland, D. C. (1962), Integral of the Bivariate Normal Distribution Over an Offset Circle, J. Amer. Statist. Assoc. 57, 758-767. (2.1.2, 2.1.3, 2.1.4)

Gilliland, D. C. (1964), A Note on the Maximization of a Non-Central Chi-Square Probability, Ann. Math. Statist. 35, 441-442. (2.1.5)

Gilliland, D. C. (1966), Some Bombing Problems, Amer. Math. Monthly 73, 713-716. (2.1.5, 2.3.4, 2.5.1)

Gilliland, D. C. (1968), On Maximization of the Integral of a Bell-Shaped Function over a Symmetric Set, Naval Research Logistics Quarterly 15, 507-515. (2.1.5)

Goodrich, R. L. (1967), Defense Models VII: Optimization of Multi-wave Attacks with Perfect Strike Reporting, Paper 10, Lambda Corporation, Arlington, Virginia. (4.7.1, 4.7.2)

Goodrich, R. L. (1968), Defense Models V: IVAN2, A Detailed Model for Preferential Defense, Paper 7, Lambda Corporation, Arlington, Virginia. (6.4)

Goodrich, R. L. (1970), Defense Models XVIII: Blotto, Extended to Arbitrary Concave Damage Laws, Paper 36, Lambda Corporation, Arlington, Virginia. (5.6.4)

Gorfinkel, M. (1963a), A Decision-Theory Approach to Missile Defense, Opns. Res. 11, 199-209. (3.4)

Gorfinkel, M. (1963b), Allocation Under Uncertainty, Handout for a talk presented at the 10th Annual International Meeting of The Institute of Management Sciences, Tokyo, Japan, August 1963. (3.4)

Grad, A. and Solomon, H. (1955), Distribution of Quadratic Forms and Some Applications, Ann. Math. Statist. 26, 464-477. (2.1.3)

Groenewoud, C., Hoaglin, D. C. and Vitalis, J. A. (1967), Bivariate Normal Offset Circle Probability Tables with Offset Ellipse Transformations, Cornell Aeronautical Laboratories, Buffalo, New York. (2.1.4)

Groves, A. D. and Smith E. S. (1957), Salvo Hit Probabilities for Offset Circular Targets, Opns. Res. 5, 222-228. (2.1.2, 2.3.2)

Grubbs, F. E. (1964a), Approximate Circular and Noncircular Offset Probabilities of Hitting, Opns. Res. 12, 51-62. (2.1.2, 2.1.3, 2.1.4)

Grubbs, F. E. (1964b), Statistical Measures of Accuracy for Riflemen and Missile Engineers, Havre De Grace, Maryland. (2.6.2)

Grubbs, F. E. (1968), Expected Target Damage for a Salvo of Rounds with Elliptical Normal Delivery and Damage Functions, Opns. Res. 16, 1021-1026. (2.4.3, 2.4.4)

Grubbs, F. E., Breaux, H. J. and Coon, H. J. (1971), Approximation Procedures and Some Key Results for Estimating Expected Target Damage, Opns. Res. 19, 645-654. (2.4.3, 2.4.4, 2.5.1)

Guenther, W. C. (1964), A Generalization of the Integral of the Circular Coverage Function, Amer. Math. Monthly 7, 278-283. (2.2.1)

Guenther, W. C. (1966), Target Coverage Problems, Proceedings of the Eleventh Conference on the Design of Experiments in Army Research Development and Testing, ARO-D Report 66-2, 573-596. (2. Introduction, 2.2.2, 2.4.3, 2.4.4)

Guenther, W. C. and Terragno, P. J. (1964), A Review of the Literature on a Class of Coverage Problems, Ann. Math. Statist. 35, 232-260. (2. Introduction, 2.4.1, 2.4.4)

Harter, H. L. (1960), Circular Error Probabilities, J. Amer. Statist. Assoc. 55, 723-731. (2.1.3)

Helgert, H. J. (1971), On the Computation of Hit Probability, Opns. Res. 19, 668-684. (2.1.6)

Holla, M. S. (1970), On a Noncentral Chi-Square Distribution in the Analysis of Weapon Systems Effectiveness, Metrika 15, 9-14. (2.3.3)

Hunter, J. J. (1967), An Analytical Technique for Urban Casualty Estimation From Multiple Nuclear Weapons, Opns. Res. 15, 1096-1108. (2.4.1, 2.5.1)

Inselmann, E. H. and Granville, W. (1967), Circular Distribution Estimation, Opns. Res. 15, 160-165. (2.6.1, 2.6.2)

Israel, J. E. (1970), Optimal Group Strategies, AR 70-1, Analytical Services Corporation, Falls Church, Virginia. (4.4.3)

Jaiswal, N. K. and Sangal, P. P. (1972), Expected Damaged Area for Stick and Triangular Pattern Bombing, Opns. Res. 20, 344-349. (2.5.1)

Jarnagin, M. P. (1965), Expected Coverage of a Circular Target by Bombs all Aimed at the Center, N.W.L. Report 1941, U.S. Naval Weapons Laboratory, Dahlgren, Virginia. (2.2.2, 2.5.2)

Jarnagin, M. P. (1966), Expected Coverage of a Circular Target by Bombs all Aimed at the Center, Opns. Res. 14, 1139-1143. (2.2.2)

Jarnagin, M. P. and DiDonato, A. R. (1965), Expected Damage to a Circular Target by a Multiple Warhead, N.W.L. Report 1936, U.S. Naval Weapons Laboratory, Dahlgren, Virginia. (2.2.2)

Jarnagin, M. P. and DiDonato, A. R. (1966), Damage to a Circular Target by a Gaussian Distribution Warhead with Uniformly Distributed Bomblets, Opns. Res. 14, 1014-1023. (2.2.2)

Kabak, J. W. (1972), On Scheduling the Delivery of Babies, Opns. Res. 20, 19-23. (3.6)

Kabe, D. G. (1965), Note on a Coverage Problem, Opns. Res. 13, 1027-1029. (2.6.1)

Kabe, D. G. (1968), Minimum Variance Unbiased Estimate of a Coverage Probability, Opns. Res. 16, 1016-1020. (2.6.1)

Kamat, A. R. (1962), Some More Estimates of Circular Probable Error, J. Amer. Statist. Assoc. 57, 191-195. (2.6.2)

Kooharian, A., Saber, N. and Young, H. (1969), A Force Effectiveness Model with Area Defense of Targets, Opns. Res. 17, 895-906. (6.2.1)

Krumm, L. G. (1969), The Optimum Use of Antiballistic Missiles for Point Target Defense, MS thesis, U.S. Naval Postgraduate School, Monterey, California. (4.5.1)

Laurent, A. G. (1957), Bombing Problems — A Statistical Approach Opns. Res. 5, 75-89. See also, Errata, Opns. Res. 6, 297. (2.6.1)

Laurent, A. G. (1962), Bombing Problems — A Statistical Approach II, Opns. Res. 10, 380-387. (2.6.1)

Lemus, F. and David, K. H. (1963), An Optimum Allocation of Different Weapons to a Target Complex, Opns. Res. 11, 787-794. (5.1)

Lilliefors, H. W. (1957), A Hand-Computation Determination of Kill Probability for Weapons Having Spherical Lethal Volume, Opns. Res. 5, 416-421. (2.1.3)

Lowe, J. R. (1960), A Table of the Integral of the Bivariate Normal Distribution Over an Offset Circle, J. Roy. Statist. Soc. Ser. B 22, 177-187. (2.1.4)

Luce, R. D. and Raiffa, H. (1957), Games and Decisions: Introduction and Critical Survey, John Wiley & Sons. (4.3.6)

Manne, A. S. (1958), A Target Assignment Problem, Opns. Res. 6, 346-351. (5.1)

Marcum, J. I. (1950), Table of Q-Functions, Rand Report RM-339, The Rand Corporation, Santa Monica, California. (2.1.2)

Marsaglia, G. (1965), Some Problems Involving Circular and Spherical Targets, Opns. Res. 13, 18-27. (2.1.5, 2.3.4, 2.5.1)

Matheson, J. D. (1966), Preferential Strategies, AR 66-2, Analytical Services Corporation, Falls Church, Virginia. (4.3.2)

Matheson, J. D. (1967), Preferential Strategies with Imperfect Weapons, AR 67-1, Analytical Services Corporation, Falls Church, Virginia. (4.3.1)

Matlin, S. (1970), A Review of the Literature on the Missile-Allocation Problem, Opns. Res. 18, 334-373. (1.3)

McGill, J. T. (1970), Solution of Singly-Constrained Concave Allocation Problems, Research Paper P-619, Institute for Defense Analyses, Arlington, Virginia. (5.1)

McNolty, F. (1962), Kill Probability When the Weapon Bias is Randomly Distributed, Opns. Res. 10, 693-701. (2.3.3)

McNolty, F. (1965), Kill Probability When the Lethal Effect is Variable, Opns. Res. 13, 478-482. (2.4.1, 2.4.2)

McNolty, F. (1967), Kill Probability for Multiple Shots, Opns. Res. 15, 165-169. (2.3.2, 2.3.3, 2.4.2)

McNolty, F. (1968a), A Note on Radial Distributions, Opns. Res. 16, 211-216. (2.3.5)

McNolty, F. (1968b), Expected Coverage for Targets of Nonuniform Density, Opns. Res. 16, 1027-1040. (2.3.5)

Miercort, F. A. and Soland, R. M. (1971), Optimal Allocation of Missiles Against Area and Point Defenses, Opns. Res. 19, 605-617. (6.2.3)

Moranda, P. B. (1959), Comparison of Estimates of Circular Probable Error, J. Amer. Statist. Assoc. 54, 794-800. See also, Corrigenda, J. Amer. Statist. Assoc. 55, 755-756. (2.6.2)

Morgenthaler, G. W. (1960), On Bernoulli Play with Limited Resources, Amer. Math. Monthly 67, 344-348. (3.5.2)

Morgenthaler, G. W. (1961), Some Circular Coverage Problems, Biometrika 48, 313-324. (2.4.1, 2.4.4)

Nadler, J. and Eilbott, J. (1971), Optimal Sequential Aim Corrections for Attacking a Stationary Point Target, Opns. Res. 19, 685-697. (2.7)

Neville, E. H. (1915), On the Solution of Numerical Functional Equations, Proc. London Math. Soc. 14 (Series 2), 308-326. (2.2.3, 2.5.1)

Oberg, E. N. (1947), Approximate Formulas for the Radii of Circles which Include a Specified Fraction of a Normal Bivariate Distribution, Ann. Math. Statist. 18, 442-447. (2.1.3)

Operations Evaluation Group (1959), Probability-of-Damage Problems of Frequent Occurrence, Study No. 626. (2.4.1, 2.4.2)

Ordway, G. L. and Rosenstock, H. M. (1963), Analytic Expressions for the Probability of Penetrating a Point Defense, Opns. Res. 11, 40-56. (3.5.2)

Owen, D. B. (1962), Handbook of Statistical Tables, Addison-Wesley, Massachusetts. (2.1.2)

Pearsall, E. S. (1971), A Lagrange Multiplier Method for Certain Constrained Min-Max Problems, Research Paper P-745, Institute for Defense Analyses, Arlington, Virginia. (5.3.1)

Pearson, K. (1934), Tables of the Incomplete Beta Function, Cambridge University Press (Second Edition, 1968). (2.4.3, 2.5.1)

Pearson, K. (1951), Tables of the Incomplete Gamma Function, Cambridge University Press. (2.5.2)

Pearson, E. S. and Hartley, H. O. (editors) (1954), Biometrika Tables for Statisticians, Volume 1, Cambridge University Press. (2.6.2)

Penn, A. I. (1967), Defense Models IX: A Variation of the Lagrangian Blotto Game, Paper 12, Lambda Corporation, Arlington, Virginia. (4.3.6)

Penn, A. I. (1969), Defense Models XV: Preferential Area Defense in Support of Preallocated Defense, Paper 27, Lambda Corporation, Arlington, Virginia. (4.4.4)

Penn, A. I. (1971), A Generalized Lagrange-Multiplier Method for Constrained Matrix Games, Opns. Res. 19, 933-945. (4.3.6, 4.4.4, 5.3.1, 5.6.4)

Perkins, F. M. (1961), Optimum Weapon Deployment for Nuclear Attack, Opns. I s. 9, 77-94. (5.4.2)

Piccariello, H. J. (1962), A Missile Allocation Problem, Opns. Res. 10, 795-798. (6.1.6)

Pugh, G. E. (1964), Lagrange Multipliers and the Optimal Allocation of Defense Resources, Opns. Res. 12, 543-567. (5.3.1, 5.4, 5.5.1, 5.7.1, 6.4)

Rand Corporation (1952), Offset Circle Probabilities, Rand Report R-234, The Rand Corporation, Santa Monica, California. (2.1.2)

Randolph, P. H. and Swinson, G. E. (1969), The Discrete Max-Min Problem, Naval Research Logistics Quarterly 16, 309-314. (5.3.2)

Read, C. B. (1971), Two Casualty-Estimation Problems Associated with Area Targets and the Circular Coverage Function, Opns. Res. 19, 1730-1741. (2.1.2, 2.4.2, 2.5.1)

Rice, S. O. (1945), Mathematical Analysis of Random Noise, Bell Syst. Tech. J. 24, 46-156. (2.1.2)

Rosenthal, G. W. and Rodden, I. J. (1961), Tables of the Integral of the Elliptical Bivariate Normal Distribution Over Offset Circles, Lockheed Report LSMD-800619, Sunnyvale, California. (2.1.4)

Ruben, M. (1960), Probability Content of Regions Under Spherical Normal Distribution, I, Ann. Math. Statist. 31, 591-618. (2. Introduction)

Sangal, P. P. (1969), Expected Coverage of a Rectangular Target for Different Bombing Strategies, Opsearch 6, 199-216. (2.5.1)

Schreiber, T. S. (1968), Defense Models XIV: Optimum Strategies in a Battle Involving Many Identical Targets, Paper 23, Lambda Corporation, Arlington, Virginia. (4.3.6)

Shapiro, M., Abramson, L. R. and Coburn, J. M. (1966), Stockpile Requirements for Battery Defense, Technical Report T-7/322, Electronics Research Laboratories, Columbia University. (6.1.1)

Smith, E. H. and Stone, D. E. (1961), A Note on the Expected Coverage of One Circle by Another, The Case of Offset Aim, Soc. Indus. Appl. Math. Rev. 3, 51-53. (2.2.1)

Solomon, H. (1953), Distribution of the Measure of a Random Two-Dimensional Set, Ann. Math. Statist. 24, 650-656. (2.1.2, 2.2.1)

Solomon, H. (1960), Distribution of Quadratic Forms, Tables and Applications, Technical Report 45, Applied Mathematics and Statistics Laboratcries, Stanford University. (2.1.3)

Svesnikov, A. A. (1948), Determination of the Best Means of Introducing Artificial Dispersion in Firing (for Various Special Cases), contained in Collection of Articles on the Theory of Firing I (A. N. Kolmogorov, ed.), Translation T-14, Rand Corporation, Santa Monica, California. (2.5.3)

Thompson, W. A. (1958), The Diffused-Target Concept, Opns. Res. 6, 671-675. (2.6.3)

van Lint, J. H. and Pollak, H. O. (1972), An "Offense-Last-Move" Game Against Perfect Local Defense at Targets of Arbitrary Values, submitted to J. of Combinatorial Theory. (5.5.2)

Walsh, J. E. (1956), Optimum Ammunition Properties for Salvos, Opns. Res. 4, 204-212. (2.5.3)

Weidlinger, P. (1962), The Lethal Radius of a Salvo, Opns. Res. 10, 547-558. (2.4)

Weiner, S. D. and McCraith, D. L. (1970), Strategies for Attacking Preferentially Defended Hard Points, Technical Report 479, Lincoln Laboratories (MIT), Lexington, Massachusetts. (6.1.5)

Weingarten, H. and DiDonato, A. R. (1961), A Table of Generalized Circular Error, Math. Comput. 15, 169-173. (2.1.3)

Young, H. A. (1965), Note on Computing Optimum Discrete Allocations, Opns. Res. 13, 449-503. (5.2.1)